Quantum error correction for continuously detected errors.

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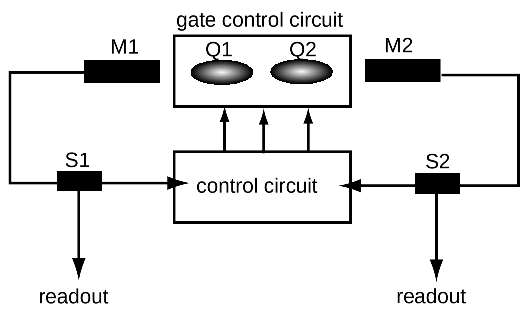
Outline.

Can we correct measurement errors ?
(eg. teleportation based schemes like linear optics QC).

If a measuring apparatus is always connected to a qubit, it may occasionally register a measurement even if it is in the quiescent state.

Can this kind of error be corrected with quantum control protocols?



Outline.

- Errors, noise and decoherence
- Measurement errors and 'spontaneous emission' errors
- Continuous conditional dynamics given a measurement record
- Quantum feedback
- Quantum feedback as error correction

See also: K. Khodjasteh and D. A. Lidar, quant-ph/0301105.

Errors, noise and decoherence.

Code states; $\{|0\rangle_L, |1\rangle_L\}$

Examples:

$$|0\rangle_L \leftrightarrow |\downarrow\ \rangle_z, |1\rangle_L \leftrightarrow |\uparrow\rangle_z$$

$$\begin{aligned} |0\rangle_L &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \\ |1\rangle_L &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle) \end{aligned}$$

Denote a generic *physical* qubit as $|0\rangle, |1\rangle$.

Errors, noise and decoherence.

Bit flip errors on physical qubits.

$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle$$

X is a Pauli operator, σ_x.

Phase flip errors on physical errors.

$$Z(|0\rangle + |1\rangle) = |0\rangle - |1\rangle$$

Z is a Pauli operator, σ_z

Errors, noise and decoherence.

Decoherence ?

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Phase error with probability p.

Average over possible events:

$$|\psi\rangle\langle\psi| \to |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1| + (1-2p)\left(\alpha\beta^*|0\rangle\langle 1| + \alpha^*\beta|1\rangle\langle 0|\right)$$

Complete decoherence when $p = 1/2$.

Errors, noise and decoherence.

Z-measurement errors.

Suppose an accidental (ideal) measurement of Z is made.

$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \begin{cases} |0\rangle & \text{with probability} \quad |\alpha|^2 \\ |1\rangle & \text{with probability} \quad |\beta|^2 \end{cases}$$

If result is not known:

$$|\psi\rangle\langle\psi| \longrightarrow |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$$

decoherence in Z basis.

Errors, noise and decoherence.

A non-ideal Z measurement error: *spontaneous emission*

Kraus measurement operator:

$$a = |0\rangle\langle 1| = \frac{1}{2}(X - iY)$$

State transformation if result is 1:

$$|\psi\rangle \rightarrow a|\psi\rangle$$

$$Prob(1) = \mathrm{tr}(\rho a^\dagger a)$$

$$\alpha|0\rangle + \beta|1\rangle \rightarrow |0\rangle \text{ with probability } |\beta|^2$$

Correction of measurement error.

Encode quantum information in correlated states:

$$|0\rangle_L = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |1\rangle_L = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

Error on first qubit is detected:

$$|\psi\rangle = \alpha|0\rangle_L + \beta|1\rangle_L \to a_1|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha|01\rangle + \beta|00\rangle)$$

Prob of error $= 1/2$.

Correction unitary

$$U_1 = \frac{1}{\sqrt{2}}(XI - ZX)$$

Continuous errors.

Errors occur at random times, with constant probability.

Define a Poisson process $dN(t)$

$$\begin{aligned} dN(t)^2 &= dN(t) \\ \mathcal{E}(dN(t)) &= \gamma dt \end{aligned}$$

Example: bit flip errors on a single physical qubit.

$$\rho(t+dt) - \rho(t) = \gamma dt X\rho(t)X + (1-\gamma dt)\rho(t) - \rho(t)$$
$$\frac{d\rho(t)}{dt} = \gamma(X\rho(t)X - \rho(t))$$

Continuous errors.

Spontaneous emission errors.

Probability of an error in time $t \to t + dt$

$$p_1(t) = \gamma \langle a^\dagger a \rangle dt$$

Define a *jump* operator: $\Omega_1 = a\sqrt{\gamma dt}$

Define *no-jump* operator: $\Omega_0 = I - \gamma a^\dagger a dt/2 - iH dt$

$$\begin{aligned} d\rho &= \Omega_0 \rho \Omega^\dagger + \Omega_1 \rho \Omega_1^\dagger - \rho \\ &= -i[H, \rho]dt + \gamma(a\rho a^\dagger - \frac{1}{2}(a^\dagger a \rho + \rho a^\dagger a))dt \end{aligned}$$

See *Wiseman, Quant. Semiclass. Optics.* **8**, *205 (1996).*

Continuous measurement, feedback and error correction.

If a measurement occurs accidentally, but we know the result, can we correct for it ?

Detected jump errors.

Two questions:

- What is the measured system state *given* a record of measured results: *conditional dynamics.*
- What is the measured system state *averaged* over all measured results: *unconditional dynamics.*

Conditional dynamics: jump-unravelling.

Given an entire history of jump times up to time t $\{[t,n] : t_1, t_2, \ldots, t_n\}$, what is the conditional state ?

Stochastic Schrödinger equation:

$$
\begin{aligned}
d|\psi_c(t)\rangle &= \Bigg[dN_c(t) \left(\frac{a}{\sqrt{\langle a^\dagger a\rangle_c(t)}} - 1 \right) + dt \\
&\quad \times \left(\frac{\langle a^\dagger a\rangle_c(t)}{2} - \frac{a^\dagger a}{2} - iH \right) \Bigg] |\psi_c(t)\rangle. \qquad (1)
\end{aligned}
$$

Average over the measurement record $\rho(t) = E[|\psi_c(t)\rangle\langle\psi_c(t)|]$, $\rho(t)$ obeys the *unconditional* master equation.

Feedback for jump detection.

Hamiltonian feedback, linear in the current:

$$H_{fb}(t) = \frac{dN(t)}{dt} V, \tag{2}$$

Feedback must act after the measurement, and average over all records:

$$\dot{\rho} = -i[H, \rho] + \mathcal{D}[e^{-iV} a]\rho. \tag{3}$$

where $\mathcal{D}[a]\rho = a\rho a^\dagger - (a^\dagger a \rho + \rho a^\dagger a)/2$

Conditional dynamics: diffusive-unravelling.

Fast jump rate $\rightarrow$ approximate Poisson process by diffusion process.

Master equation is invariant under:

$$\begin{aligned} a &\rightarrow a + \alpha \\ H &\rightarrow H - \frac{i|\alpha|}{2}(e^{-i\phi} a - e^{i\phi} a^\dagger) \end{aligned}$$

In δt number of detections $\delta N(t) \approx |\gamma|^2 \delta t \gg 1$ is very large,

$$\delta N(t) \approx |\gamma|^2 \delta t + |\gamma| \langle e^{-i\phi} a + a^\dagger e^{i\phi} \rangle_c \delta t + |\gamma| \delta W(t), \tag{4}$$

$\delta W(t)$ Gaussian: mean zero, variance δt.

Conditional dynamics: diffusive-unravelling.

Define the stochastic measurement record:

$$\begin{aligned}\frac{dQ(t)}{dt} &= \lim_{\gamma\to\infty}\frac{\delta N(t)-|\gamma|^2\delta t}{|\gamma|\delta t}\\ &= \langle e^{-i\phi}a+e^{i\phi}a^\dagger\rangle_c + dW(t)/dt.\end{aligned}$$

Conditional state, given current record:

$$\begin{aligned}d\rho_c(t) &= -i\,[H,\,\rho_c(t)]\,dt+\mathcal{D}[e^{-i\phi}a]\rho_c(t)dt\\ &\quad +\mathcal{H}[e^{-i\phi}a]\rho_c(t)dW(t).\end{aligned} \tag{5}$$

$\langle a\rangle_c = \mathrm{tr}\,(\rho_c a)$, dW *Wiener increment*), and

$$\mathcal{H}[a]\rho = a\rho+\rho a^\dagger-\rho\,\mathrm{tr}\,[a\rho+\rho a^\dagger].$$

Feedback: diffusive-unravelling.

Feedback hamiltonian:

$$H_{fb}(t) = \frac{dQ(t)}{dt} F, \tag{6}$$

Average over all current records;

$$\begin{aligned} \dot{\rho} &= -i[(e^{i\phi} a^\dagger F + e^{-i\phi} F a)/2 + H, \rho] \\ &\quad + \mathcal{D}[e^{-i\phi} a - iF]\rho \end{aligned} \tag{7}$$

Jump correction code.

Codewords:

$$
\begin{aligned}
|0\rangle_L &\equiv (|00\rangle + |11\rangle)/\sqrt{2} \\
|1\rangle_L &\equiv (|01\rangle + |10\rangle)/\sqrt{2}.
\end{aligned}
$$

(Stabilizer generator XX)

Error on first qubit,
$|0\rangle_L \rightarrow |01\rangle$ and $|1\rangle_L \rightarrow |00\rangle$,

Now apply a unitary transformations to correct:

$$
\begin{aligned}
U_1 &= (XI - ZX)/\sqrt{2} \\
U_2 &= (IX - XZ)/\sqrt{2}.
\end{aligned}
$$

Continuous correction by feedback.

The jump operators,

$$\Omega_j = \sqrt{\kappa_j dt}(X_j - iY_j),$$

$$\dot{\rho} = \sum_{j=1,2} \kappa_j \mathcal{D}[X_j - iY_j]\rho - i[H, \rho].$$

No-jump operation;

$$\begin{aligned} \Omega_0 &= II\,(1 - (\kappa_1 + \kappa_2)dt) - \kappa_1 dt ZI \\ &\quad -\kappa_2 dt IZ - iH dt. \end{aligned}$$

Continuous correction by feedback.

Correct no-jump term by adding a driving Hamiltonian:

$$H = -(\kappa_1 YX + \kappa_2 XY).$$

Then,

$$\begin{aligned}\Omega_0 &= II\,(1-(\kappa_1+\kappa_2)dt) - \kappa_1 dt ZI(II - XX) \\ &\quad - \kappa_2 dt IZ(II - XX),\end{aligned}$$

$II - XX$ acts to annihilate the codespace, Ω_0 acts trivially on the codespace.

Jump correction code.

With the driving hamiltonian,

$$d\rho = \Omega_0 \rho \Omega_0^\dagger - \rho + dt \sum_{j=\{1,2\}} \kappa_j U_j a_j \rho a_j^\dagger U_j^\dagger,$$

where U_j is the recovery operator

$$H_{fb} = \sum_{j=1,2} \frac{dN_j(t)}{dt} V_j, \tag{8}$$

$V_j \propto U_j$. On code space: $U_j a_j \rho a_j^\dagger U_j^\dagger =_c I$

The master equation preserves the code space.

Diffusion correction code.

Feedback operators:

$$\begin{aligned} F_1 &= \sqrt{\kappa_1}(XI - ZX) \\ F_2 &= \sqrt{\kappa_2}(IX - XZ). \end{aligned}$$

Choose same driving Hamiltonian as Jump case.

$$\dot{\rho} = \kappa_1 \mathcal{D}[YI - iZX]\rho + \kappa_2 \mathcal{D}[IY - iXZ]\rho$$

$YI - iZX = YI(II - XX)$, and $IY - iXZ = IY(II - XX)$.

$II - XX$ annihilates the codespace.

Conclusion.

- Spontaneous emission errors can be corrected continuously if they are detected.
- Spontaneous emission errors correspond to accidental non-ideal Z measurement errors.
- Feedback can correct for measurement errors.
- Also applies to ideal measurement errors.
- Can be generalized to n qubits with encoded operations (see Ahn et al., quant-ph/0302006).