Optimal design of the experiment of the century

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+ Wim van Dam, Peter Grünwald

Quantum vs. classical (=local realist) *theory*:

QM vs. LR

EPR: Einstein, Podolsky, Rosen (1935):

QM is incomplete, if true (assuming LR)

Bell (1964):

LR is false, if QM true

Same, but now with proposed *experiment*:

CHSH: Clauser, Horne, Shimony, Holt (1969)

GHZ: Greenberger, Horne, Zeilinger (1989)

Hardy: Hardy (1993)

Bell: Bell (1964)

Mermin: Mermin (1985)

Acín: Acín, Durt, Gisin, Latorre (2002)

CHSH

Innsbruck, 1998

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http://www.quantum.at
     ---> people -> Gregor Weihs -> Bell experiment
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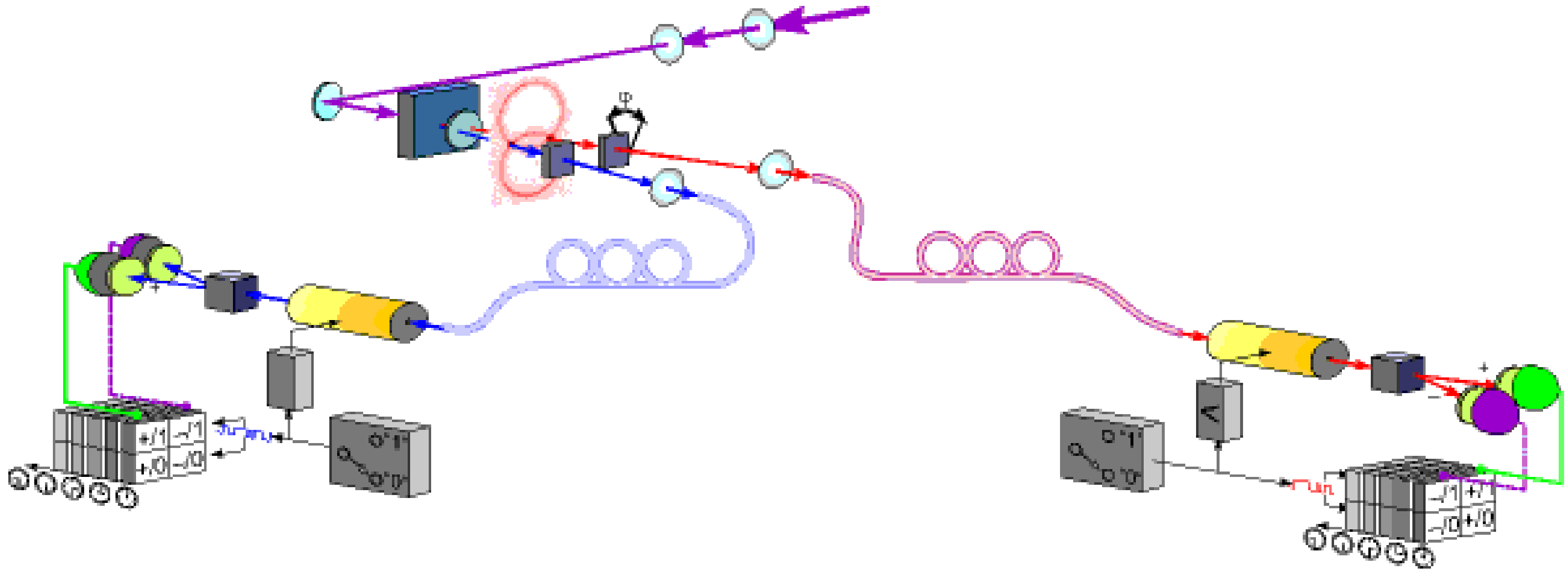
CHSH



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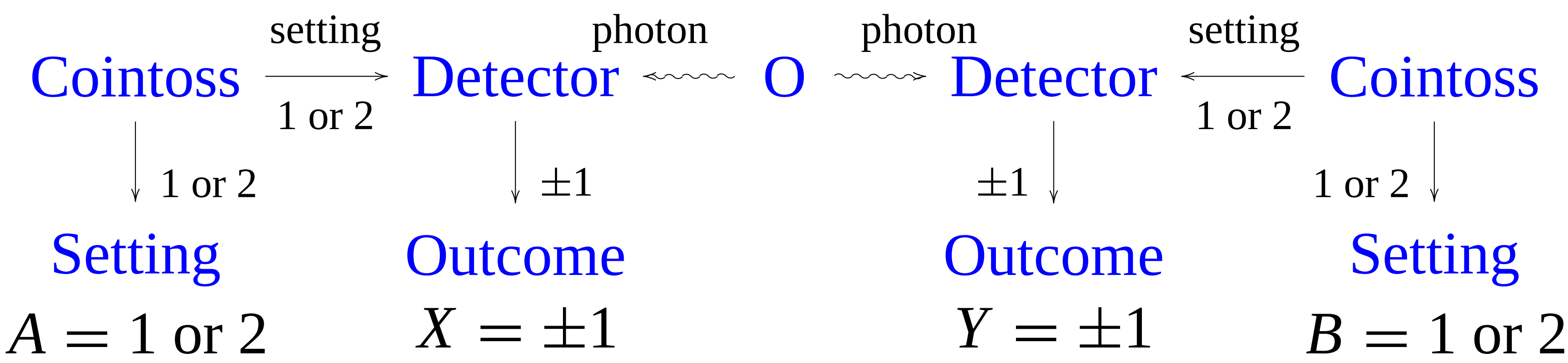
CHSH



$Q^{\psi\cdots}_{ab}$: probability distribution of (X, Y) given $A = a$, $B = b$

$$Q^{\psi\cdots}_{ab}(x, y) = Q^{\psi\cdots}_{ab}\{X = x, Y = y\}$$

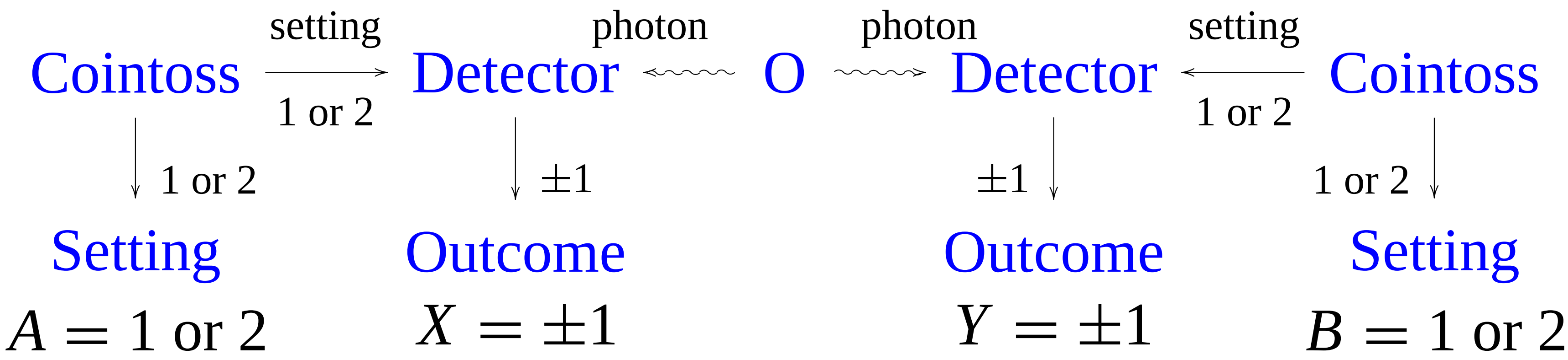
σ : probability distribution of (A, B)

$$\sigma(a, b) = \sigma\{A = a, B = b\}$$

$Q^{\sigma\psi\cdots}$: probability distribution of (A, B, X, Y)

$$Q^{\sigma\psi\cdots}(a, b, x, y) = \sigma(a, b) Q^{\psi\cdots}_{ab}(x, y)$$

CHSH



CHSH specifies $\psi \ldots$:

$$Q_{ab}^{\psi\cdots}\{X = \pm 1\} \;=\; Q_{ab}^{\psi\cdots}\{Y = \pm 1\} \;=\; \tfrac{1}{2}$$

$$Q_{12}^{\psi\cdots}\{X = Y\} \;=\; \tfrac{1}{2}(1 + 1/\sqrt{2}) \;=\; 0.85$$

$$Q_{ab}^{\psi\cdots}\{X = Y\} \;=\; \tfrac{1}{2}(1 - 1/\sqrt{2}) \;=\; 0.15 \quad (ab) \neq (12)$$

Weihs chose σ uniform:

$$\sigma : \quad A \perp B, \quad A, B \sim \text{Bernoulli}(\tfrac{1}{2})$$

CHSH

			Bob			
			1 , +	1 , –	2 , +	2 , –
		Zählungen	104122	100144	93348	90841
Alice	1 , +	77988	313	1728	1636	179
	1 , –	74935	1978	351	294	1143
	2 , –	75892	418	1683	269	1100
	2 , –	73456	1578	361	1386	156

Koinzidenzen

			QuantentheoretischeVorhersage		
Zeitverschiebung	3,8	ns			
Koinzidenzfenster	4,0	ns			
Koinzidenzengesamt	14573		ideal	bei 94% -97% Kontrast	
ρ(1,1)	-0,70	± 0,01	-0,71	0,68	± 0,02
ρ(2,1)	-0,61	± 0,01	-0,71	0,68	± 0,02
ρ(1,2)	0,71	± 0,01	0,71	0,68	± 0,02
ρ(2,2)	-0,71	± 0,01	-0,71	0,68	± 0,02
S	2,73	± 0,02	2,82	2,72	± 0,04
Verletzung	29,8	stddev			

Tabelle 5.3: *Auswertung von longdist35, aufgenommen am 22.4.1998*

Local Realism

$$\exists\ X_1, X_2, Y_1, Y_2\ :\quad X \equiv X_A, \quad Y \equiv Y_B$$

$$(A, B) \perp (X_1, X_2, Y_1, Y_2) \sim \pi$$

P_{ab}^{π}: probability distribution of (X, Y) given $A = a$, $B = b$

$$P_{ab}^{\pi}(x, y) = P_{ab}^{\pi}\{X = x, Y = y\}$$

σ: probability distribution of (A, B)

$$\sigma(a, b) = \sigma\{A = a, B = b\}$$

$P^{\sigma\pi}$: probability distribution of (A, B, X, Y)

$$P^{\sigma\pi}(a, b, x, y) = \sigma(a, b) P_{ab}^{\pi}(x, y)$$

Local Realism

$$\exists \ X_1, X_2, Y_1, Y_2 \ : \quad X \equiv X_A, \quad Y \equiv Y_B$$
$$(A, B) \perp (X_1, X_2, Y_1, Y_2) \sim \pi$$

P^{π}_{ab}: probability distribution of (X, Y) given $A = a$, $B = b$

π: probability distribution of (X_1, X_2, Y_1, Y_2), margins π_{ab}

$$\begin{aligned} P^{\pi}_{ab}(x, y) &= P^{\pi}_{ab}\{X = x, Y = y\} \\ &= \pi\{X_a = x, Y_b = y\} = \pi_{ab}(x, y) \\ &= \sum_{x_1, x_2, y_1, y_2 \,:\, x_a = x, y_b = y} \pi(x_1, x_2, y_1, y_2) \end{aligned}$$

Bell's Theorem

Fix σ, define
$\mathscr{P}^\sigma = \{P^{\sigma\pi} : \pi \text{ arbitrary}\}$, $\mathscr{Q}^\sigma = \{Q^{\sigma\psi} : \psi \ldots \text{ arbitrary}\}$.

$\mathscr{P}^\sigma$ and $\mathscr{Q}^\sigma$ are compact and convex, $\mathscr{P}^\sigma \subseteq \mathscr{Q}^\sigma$

$\mathscr{Q}^\sigma \setminus \mathscr{P}^\sigma \neq \emptyset$

CHSH proof: specific choice of $Q^\psi_{ab}(x, y)$

Hardy proof: specific choice of $Q^\psi_{ab}(x, y)$

GHZ proof: three parties, specific choice of $Q^\psi_{abc}(x, y, z)$

In *experiment* must also choose $\sigma(a, b)$, or $\sigma(a, b, c)$:
fair or biased coins, uncorrelated or correlated

Bell's Theorem

CHSH:

$$\pi\{X_1 = Y_2\} - \pi\{X_1 = Y_1\} - \pi\{X_2 = Y_1\} - \pi\{X_2 = Y_2\} \leq 0$$

hence

$$P^{\pi}_{12}\{X = Y\} - P^{\pi}_{11}\{X = Y\} - P^{\pi}_{21}\{X = Y\} - P^{\pi}_{22}\{X = Y\} \leq 0$$

but under QM we can have

$$Q^{\psi}_{12}\{X = Y\} - Q^{\psi}_{11}\{X = Y\} - Q^{\psi}_{21}\{X = Y\} - Q^{\psi}_{22}\{X = Y\} = \sqrt{2} - 1 > 0$$

Bell's Theorem

Hardy:

$\pi_{11}(--) = 0$ and $\pi_{12}(+-) = 0$ and $\pi_{21}(-+) = 0$ implies
$\pi_{22}(--) = 0$

hence
$P^{\pi}_{11}(--) = 0$ and $P^{\pi}_{12}(+-) = 0$ and $P^{\pi}_{21}(-+) = 0$ implies
$P^{\pi}_{22}(--) = 0$

but under QM we can have
$Q^{\psi}_{11}(--) = 0$ and $Q^{\psi}_{12}(-+) = 0$ and $Q^{\psi}_{21}(+-) = 0$ *and*
$Q^{\psi}_{22}(--) = 0.09$

Bell's Theorem

GHZ:

$$\pi\{X_2 Y_1 Z_1 = +\} = \pi\{X_1 Y_2 Z_1 = +\} = \pi\{X_1 Y_1 Z_2 = +\} = 1$$
implies $\pi\{X_2 Y_2 Z_2 = +\} = 1$

hence

$$P^{\pi}_{211}\{XYZ = +\} = P^{\pi}_{121}\{XYZ = +\} = P^{\pi}_{112}\{XYZ = +\} = 1$$
implies $P^{\pi}_{222}\{XYZ = +\} = 1$

but under QM we can have

$$Q^{\psi}_{211}\{XYZ = +\} = Q^{\psi}_{121}\{XYZ = +\} = Q^{\psi}_{112}\{XYZ = +\} = 1$$
and $Q^{\psi}_{222}\{XYZ = +\} = 0$

Quantum Mechanics (von Neumann)

ρ, $\widehat{X}$ operators on Hilbert space $\mathfrak{H}$, $\psi \in \mathfrak{H}$; $f : \mathbb{R} \to \mathbb{R}$:

ρ is a *state* if $\rho \geq 0$, $\mathrm{trace}(\rho) = 1$
$\rho = \int \Pi_{[\psi]} P(\mathrm{d}\psi)$ (mixture of pure states)

$\widehat{X}$ is an *observable* if $\widehat{X}^* = \widehat{X}$
$f(\widehat{X})$ too is an observable
(replace eigenvalues by f of eigenvalues) e.g. $\mathbb{1}_A\{\widehat{X}\}$

Experiment: *measure* observable $\widehat{X}$ on system in state ρ
Outcome is random, call it X

$Q^{\rho,\widehat{X}}\{X \in A\} = \mathrm{trace}(\rho \mathbb{1}_A(\widehat{X}))$
$\mathrm{E}_{\mathrm{Q}}(f(X)) = \mathrm{trace}(\rho f(\widehat{X}))$

Quantum Mechanics

Commuting observables can be measured simultaneously

$$Q^{\rho,\widehat{X}}\{X \in A, Y \in B\} = \mathrm{trace}(\rho \mathbb{1}_A(\widehat{X})\mathbb{1}_B(\widehat{Y}))$$

Product space $\mathfrak{H} \otimes \mathfrak{K}$

Joint state ρ on $\mathfrak{H} \otimes \mathfrak{K}$

Observables $\widehat{X}$ on $\mathfrak{H}$, $\widehat{Y}$ on $\mathfrak{K}$ define commuting observables

$$\widehat{X} \otimes 1,\ 1 \otimes \widehat{Y} \text{ on } \mathfrak{H} \otimes \mathfrak{K}$$

Joint measurement of $\widehat{X}$ on $\mathfrak{H}$ with $\widehat{Y}$ on $\mathfrak{K}$:

$$Q^{\rho,\widehat{X},\widehat{Y}}\{X \in A, Y \in B\} = \mathrm{trace}(\rho \mathbb{1}_A(\widehat{X}) \otimes \mathbb{1}_B(\widehat{Y}))$$

CHSH

$\mathfrak{H} = \mathfrak{K} = \mathbb{C}^2$

$|☺\rangle, |☹\rangle$ orthonormal basis of $\mathbb{C}^2$

$$\psi \;=\; |☺\rangle \otimes |☺\rangle + |☹\rangle \otimes |☹\rangle$$

$$\rho = \Pi_{[\psi]}$$

$\widehat{X}_\alpha$: eigenvalues ± 1,

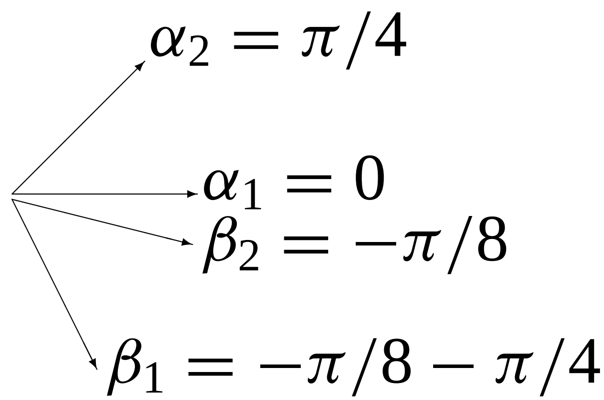
eigenvectors $\cos\alpha\,|☺\rangle + \sin\alpha\,|☹\rangle$, $\sin\alpha\,|☺\rangle - \cos\alpha\,|☹\rangle$

$\widehat{Y}_\beta$: ...

$$Q^{\rho,\widehat{X}_\alpha,\widehat{Y}_\beta}\{X = \pm 1\} = \tfrac{1}{2} = Q^{\rho,\widehat{X}_\alpha,\widehat{Y}_\beta}\{Y = \pm 1\}$$

$$Q^{\rho,\widehat{X}_\alpha,\widehat{Y}_\beta}\{X = Y\} = \cos^2\bigl(\alpha - \beta\bigr)$$

CHSH



$$\alpha_1 - \beta_2 = \pi/8$$

$$\alpha_a - \beta_b = \pi/2 \pm \pi/8 \quad \text{for } ab \neq 12$$

$$\cos^2(\alpha_1 - \beta_2) = 0.85$$

$$\cos^2(\alpha_a - \beta_b) = 0.15 \quad \text{for } ab \neq 12$$

Given experiment, which σ is best? Which experiment is best?

Fix experiment: $Q^{\psi}_{ab}(x, y)$
Experimenter chooses σ
Repeat N times (large), collect data $\sim Q^{\sigma\psi}$
Local realist must explain with π

Bayesian data analyst:

$$-\log(\Pr\{\mathcal{P}^{\sigma}|\text{data}\})/N \approx \inf_{\pi} D(Q^{\sigma\psi} : P^{\sigma\pi})$$

Frequentist data analyst:
p-value of best test statistic of $\mathrm{H}_0 : \mathcal{P}^{\sigma}$ vs. $\mathrm{H}_1 : \mathcal{Q}^{\sigma}$

$$-\log(p(\text{data}))/N \approx \inf_{\pi} D(Q^{\sigma\psi} : P^{\sigma\pi})$$

Computer scientist: Increased description length for data $\sim Q^{\sigma\psi}$
when using best coding for $\mathcal{P}^{\sigma}$

$D(Q:P) \geq 0$ *relative entropy* (*Kullback-Leibler divergence*)
of P relative to (from) Q

$$
\begin{aligned}
D(Q:P) &= \mathrm{E}_Q \log \frac{\mathrm{d}Q}{\mathrm{d}P} \\
&= \sum_{abxy} Q(a,b,x,y) \log \frac{Q(a,b,x,y)}{P(a,b,x,y)} \\
&= \sum_{ab} \sigma(a,b) D(Q^{\psi}_{ab} : P^{\pi}_{ab})
\end{aligned}
$$

Problem: how to compute

$$
\sup_{\sigma} \inf_{\pi} \sum_{ab} \sigma(a,b) D(Q^{\psi}_{ab} : P^{\pi}_{ab})
$$

Missing Data

Fix $\sigma, \psi, \ldots$

Computation of

$$\inf_{\pi} \mathrm{E}_Q \log \frac{\mathrm{d}Q}{\mathrm{d}P^{\pi}}$$

is computation of the *maximum likelihood estimator of* π given observations on (A, B, X, Y) with cell frequencies equal to their expectations under $Q^{\sigma\psi}$

The problem is a *nonparametric missing data problem*:
observe (A, B, X, Y)

function of (A, B) and (X_1, X_2, Y_1, Y_2)

$(A, B) \perp (X_1, X_2, Y_1, Y_2)$

$(X_1, X_2, Y_1, Y_2) \sim \pi$ completely unknown

Use VD-IQM; EM too slow.

Game Theory (von Neumann)

Define $D(\sigma, \pi) = D(Q^{\sigma\psi} : P^{\sigma\pi})$

$$= \sum_{ab} \sigma(a, b) D(Q^{\psi}_{ab} : P^{\pi}_{ab}) = \sum_{ab} \sigma(a, b) D_{ab}(\pi)$$

$$\sup_{\sigma} \inf_{\pi} D(\sigma, \pi) \ \leq \ \inf_{\pi} \sup_{\sigma} D(\sigma, \pi)$$

The *game has a value* when equality holds

The *game has a saddlepoint* when $\exists\, \sigma^*, \pi^*$ such that

$$\sigma^* \ = \ ! \max_{\sigma} D(\sigma, \pi^*)$$

$$\pi^* \ = \ ! \min_{\pi} D(\sigma^*, \pi)$$

Game Theory

Proposition: Game has saddle point implies game has value

Typically no value, no saddle point

Von Neumann: typically, *randomized game* has saddle point

Game Theory

$D(\sigma, \pi)$ is linear in σ, convex in π

(nonparametric missing data problem)

If σ is arbitrary, randomizing over σ doesn't change the game!

Game randomized in π has saddle point

By convexity in π, saddle point $\widetilde{\pi}^*$ is non-randomized π^*

By linearity in σ, saddle point π^* is an equalizer for $D_{ab}(\pi)$

If σ is arbitrary, we need just search for σ^*, π^* such that

$D_{ab}(\pi^*)$ is constant in (a, b)

π^* is npmle for π based on $Q^{\sigma^*\psi}(a, b, x, y)$

Results

CHSH: 0.0463 “bits per trial”; σ uniform is optimal

Hardy: 0.0280 (uncorrelated 0.0279, uniform 0.0278)

GHZ: 0.40 (uniform 0.20)

Mermin: 0.021 (uniform 0.016)

Bell: 0.016 (uncorrelated 0.015, uniform 0.013)

Acín trinary-outcome: 0.058 (maximally entangled),
0.072 (maximally resistant).

Gill trinary outcome: 0.0768

Conjecture: CHSH is best two-party binary-outcome experiment
GHZ best three-party binary-outcome

Adverts

`http://www.math.uu.nl/people/gill -->`

Time, Finite Statistics and Bell's Fifth Position
quant-ph/0301059

Martingale inequalities and stochastic integration
to design protocol of bet with Luigi Accardi

Invitation to Quantum Tomography, with Madalin Guta
quant-ph/0303020

Quantum metric entropy with bracketing...

Adverts

Aarhus, August 2003 — workshop
`www.maphysto.dk`

Berlin, August 2003, ISI conference — invited paper session on quantum statistical information `www.isi-2003.de`