Quantum Field Theory in Curved Spacetime

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Quantum Field Theory in Curved Spacetime

Quantum field theory in curved spacetime is the theory of a quantum field propagating in a classical (globally hyperbolic) curved spacetime (M, g_{ab}). One can consider "back-reaction" effects within the context of this theory by imposing the semiclassical Einstein equation $G_{ab} = 8\pi \langle T_{ab} \rangle_\omega$, but those effects will not be considered here: This talk will focus exclusively on the formulation of interacting quantum field theory in curved spacetime (at the perturbative level). I will restrict consideration to a scalar field for simplicity, but the results should be generally applicable to all quantum fields.

Flat vs. Curved Spacetime

Usual formulations of QFT in flat spacetime make crucial use of:

- Poincare symmetry
- A preferred (Poincare invariant) vacuum
- Notion of "particles"
- Momentum space methods
- S-matrices
- Euclidean methods

These features will be absent in general curved spacetimes. How does one formulate QFT? (Is QFT compatible with "general covariance"?)

Phenomena in QFTCS have, thus far, provided our deepest insights into the nature of quantum gravity. It is my hope (& expectation) that the proper formulation of QFTCS will also provide deep insights into quantum gravity — even though gravity itself is being treated classically.

Linear (Free) Quantum Fields in Curved Spacetime

The fundamental commutation relations for a free scalar field ϕ generalize straightforwardly to curved spacetime:

$$[\phi(f), \phi(g)] = -i \underbrace{\Delta}_{\text{adv - ret Greens fun.}}(f,g)\, 1$$

However, in the absence of time translation symmetry, have no preferred "vacuum state" nor any natural particle interpretation of the theory. Worse yet, in general, there is no preferred Hilbert space representation of the canonical commutation relations. Depending upon the asymptotics of the spacetime, the S-matrix cannot, in general, be naturally defined, and, when defined, need not exist.

Solution to all difficulties: Formulate the theory via the algebraic approach. Make all physical predictions in terms of probabilities for measuring field observables in given states.

Some Details of Linear QFTCS

Quantum fields make sense only distributionally, so wish to define an algebra, $\mathcal{A}$, of observables generated by expressions of form $\phi(f)$, where $f \in C_0^\infty$, i.e. f is smooth & of compact support.

Consider the "free algebra" $\mathcal{A}_0$ composed of all finite linear combinations of finite products of ϕ's & ϕ^*'s, e.g. expressions like

$$c_1\,\phi(f_1)\,\phi(f_2) + c_2\,\phi^*(f_3)\,\phi^*(f_4)\,\phi(f_5)$$

Impose: i) Linearity of $\phi(f)$ in f

ii) $\phi^*(f) = \phi(\bar{f})$ complex conjugate

iii) $\phi([\nabla_a \nabla^a - \xi R - m^2] f) = 0$

iv) $[\phi(f), \phi(g)] = -i\,\Delta(f,g)\,1$ adv.-ret. solution

The desired algebra, $\mathcal{A}$, is defined to be $\mathcal{A}_0$ "factored" by the above relations.

States are maps $\omega : \mathcal{A} \to \mathbb{C}$ satisfying $\omega(A^*A) \geq 0$. For $A \in \mathcal{A}$, $\omega(A)$ is interpreted as the expectation value of A in the state ω. The probability distribution for A in the state ω can be obtained by going to any Hilbert space rep. that includes ω (such as the GNS rep. of ω) and computing by usual Hilbert space methods (provided that the observables are represented as self-adjoint ops.)

This yields a completely satisfactory theory of a linear quantum field in curved spacetime as far as observables in $\mathcal{A}$ are concerned, i.e., essentially the n-point functions of ϕ.

Going Beyond the Theory of a Linear Field Based Upon $\mathcal{A}$

- Even if one were only interested in a linear field, there are many observables of interest (such as T_{ab}) than merely the ones found in $\mathcal{A}$ (i.e., the "n-point functions"). We need an enlarged algebra of observables.
- To make sense of nonlinear fields, one clearly must make sense of nonlinear functions of a quantum field. Perturbative rules for constructing interacting QFT require that one define "Wick powers" of a free field as well as the time-ordered-products of these Wick powers. Again, we need an enlarged algebra of observables.

The basic difficulties: (i) $\phi(x)$ is an algebra-valued distribution, so, a priori, $[\phi(x)]^2$ does not make sense. Attempts to define $\phi^2(f) = \lim \int \phi(x)\,\phi(y)\, f(x)\, F_n(x,y)$ as $F_n(x,y) \to \delta(x,y)$ yield divergent results, so some "regularization" is needed.

(ii) Once ϕ^k is defined as an algebra-valued distribution, $T\left(\phi^{k_1}(x_1) \ldots \phi^{k_n}(x_n)\right)$ can be straightforwardly defined by "time ordering" when the support properties of $f_1, \ldots, f_n$ have suitable causal relations. We must extend this distribution to act on all test functions. The difficult part is to extend the distribution to the "total diagonal" $x_1 = x_2 = \ldots = x_n$.

"Microlocal Analysis"

[Hormander, Wightman, Radzikowski, Fredenhagen, ...]

Restrict attention to distributions, Ψ, of compact support. (If necessary, consider $f\Psi(g) \equiv \Psi(fg)$ where $f \in C_0^\infty$.) May pretend that supp Ψ is embedded in $\mathbb{R}^n$. (Must then later check the "coordinate invariance" of the various constructions & results.)

Define

$$\hat{\Psi}(k) = \frac{1}{(2\pi)^{n/2}} \Psi(e^{ik_\mu x^\mu})$$

Then $\hat{\Psi}(k)$ is a polynomially bounded analytic function of k.

Two further key results hold in the case where Ψ corresponds to a smooth function $\psi \in C_0^\infty$, i.e. $\Psi(f) = \int \psi f$:

1) $\hat{\Psi}(k) \to 0$ as $|k| \to \infty$ faster than any inverse power of $|k|$.

2) We have

$$\psi^2(x) = \frac{1}{(2\pi)^n} \int \hat{\psi}(k) \hat{\psi}(K-k) e^{iK\cdot x} dk dK$$

i.e. ψ^2 is the Fourier transform of the function $F(K) = \frac{1}{(2\pi)^{n/2}} \int \hat{\psi}(k) \hat{\psi}(K-k) dk$

The Wavefront Set

Let Φ be a distribution of compact support. Let $(x, k) \in T_*(M)$, with $k \neq 0$. Call (x, k) a nonsingular point/direction of Φ if there exists an $f \in C_0^\infty$ with $f(x) \neq 0$ and there exists an open nd. $\mathcal{O}$ of k s.t. for all $k' \in \mathcal{O}$, and all n, $\exists$ C_n s.t. $\widehat{(f\Phi)}(\lambda k') \leq \frac{C_n}{\lambda^n + 1}$ for all $\lambda > 0$.

Define

$$WF(\Phi) = \left\{ (x, k) \in T_*(M) \,\middle|\, k \neq 0,\ (x, k) \text{ is not a nonsingular point/direction of } \Phi \right\}$$

This gives a refined characterization of the singularities of distributions. Its main advantage is that it allows one to define products of distributions if their wavefront set properties are such that the Fourier convolution integral converges.

Enlargement of the Algebra of Observables

Expect Wick powers to be defined only for a restricted class of states on $\mathcal{A}$.

Hadamard states:

$$\langle \phi(x)\phi(y)\rangle_\omega = \frac{\Delta^{1/2}(x,y)}{\sigma} + V \ln\sigma + W$$

σ ← squared geodesic distance between x & y

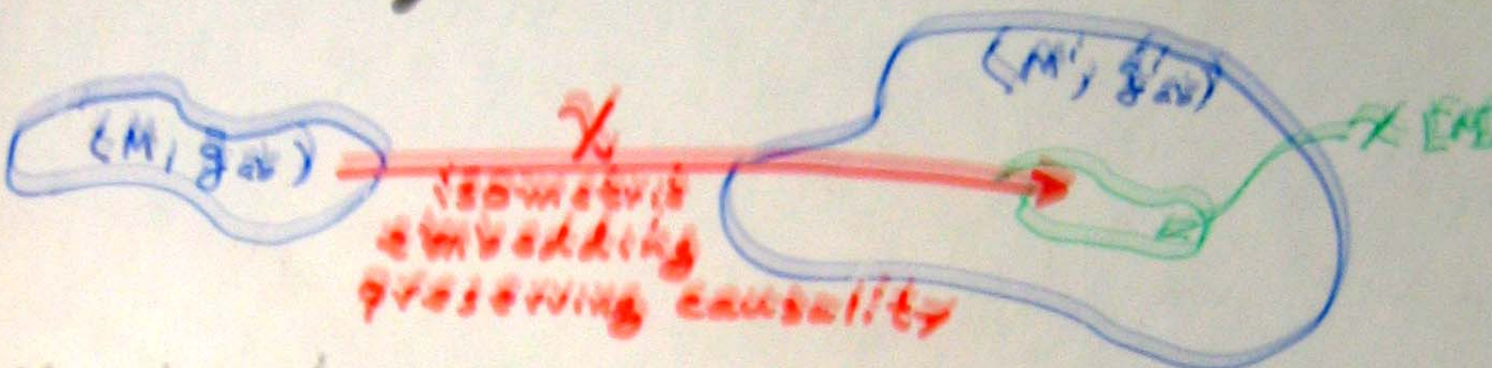
Radzikowski: Hadamard condition is equivalent to a simple condition on $WF\left[\langle\phi(x)\phi(y)\rangle_\omega\right]$.

BFK Construction of enlarged algebra of observables $\mathcal{W}$: Choose a quasi-free ("vacuum") Hadamard state ω on $\mathcal{A}$. In the GNS rep. of ω ($\equiv$ Fock space based on the vacuum state ω), consider the normal ordered n-point functions $:\phi(x_1)\cdots\phi(x_n):_\omega$. The wavefront set properties of these operator-valued-distributions are such that, for a dense set of vectors, they continue to make sense as operators when smeared with a wide class of distributions, including $f(x_1)\,\delta(x_1,\ldots,x_n)$. The resulting algebra of operators, $\mathcal{W}$, is independent of the choice of ω and defines a suitable enlarged algebra of observables. (Can also define $\mathcal{W}$ abstractly.)

However, the labeling of elements as $:\phi^n:_\omega$ **does** depend on ω. Which elements of the algebra should be viewed as representing the "true" $\phi^2(f)$ or $T(\phi^4(f)\,\phi^6(g))$, etc.?

Local, Covariant Fields

The algebra $\mathcal{W}$ depends upon the spacetime (M, g_{ab}); the algebras for different spacetimes cannot, in general, be "compared". However, suppose that one has the following situation:



Then χ gives rise to a natural injective $*$-homomorphism

$$\iota_\chi : \mathcal{W}[M, g_{ab}] \to \mathcal{W}[M', g'_{ab}]$$

A quantum field, Φ, is an assignment to every globally hyperbolic spacetime (M, g_{ab}) a distribution $\Phi_{(M, g_{ab})}$ valued in $\mathcal{W}[M, g_{ab}]$. Φ is said to be local and covariant if whenever $\chi : M \to M'$ is a causality preserving isometric embedding we have for any test function f on $\chi[M]$,

$$\iota_\chi\left(\Phi_{(M,g)}(f \circ \chi)\right) = \Phi_{(M', g')}(f)$$

Axioms for Wick Powers

1) ϕ^n should be "local and covariant"

2) $[\phi^n(x), \phi(y)] = i n \Delta(x,y) \phi^{n-1}(x)$

3) $(\phi^n(f))^* = \phi^n(\bar{f})$

4) For any quasi-free Hadamard state ω, $\omega(\phi^n(x))$ is smooth.

5) ϕ^n varies analytically (smoothly) under an analytic (smooth) variation of the metric and coupling parameters. ← (Defined in terms of a wavefront set condition on $\phi^n[g_{ab}(s)](x)$, viewed as a distribution on $\mathbb{R} \times M$.)

6) Under scaling of the metric, $g_{ab} \to \lambda^2 g_{ab}$, have $\phi^n \longrightarrow \phi^n(\lambda)$ where

$$\lambda^{-d} \phi^n(\lambda) = \phi^n + \sum_{i=1}^{m} \ln^i \lambda \, \psi_i$$

The axioms for time ordered products of Wick powers are similar, except that (4) is replaced by a much more complicated "microlocal spectral condition" and their are additional "unitarity" and "causal factorization" conditions.

Sketch of Uniqueness Argument

Consider ϕ^2. The commutation condition (2)

$$[\phi^2(x), \phi(y)] = 2i\,\Delta(x,y)\,\phi(x)$$

uniquely determines ϕ^2 up to a multiple of the identity, i.e., up to a term of the form $C(x)\,1$. The local covariance condition $\Rightarrow$ C at x depends only on the spacetime geometry in an arbitrarily small nd. of x. Continuity, analyticity, & scaling $\Rightarrow$ C is a local curvature term of the "correct dimension", in this case $C = \alpha R$.

By induction, each higher power of ϕ leads to a new "multiple of the identity" ambiguity, given by curvature terms of the appropriate dimension.

A similar — but much more complicated — argument for time-ordered-products yields uniqueness up to certain specified ambiguities. These ambiguities in the def. of time ordered products give rise to corresponding ambiguities in the definition of ϕ_I. These latter ambiguities correspond precisely to adding appropriate "counterterms" to the Lagrangian $\mathcal{L}$, but these counterterms now include curvature couplings.

Existence of Wick Powers

It is well known how to define quantities quadratic in ϕ (such as ϕ^2 or $T_{\mu\nu}$) by an appropriate "point-splitting" prescription. Let

$$:\phi(x)\,\phi(y):_H \;=\; \phi(x)\,\phi(y) - H(x,y)\,\mathbb{1}$$

locally constructed Hadamard parametrix

Define $\phi^2(f)$ to be $:\phi(x)\phi(y):_H$ smeared with $f(x)\,\delta(x,y)$.

Can define $\phi^n(f)$ by a suitable "combinatorical generalization" of this prescription.

Note that the definition of Wick powers involves the subtraction of locally and covariantly constructed distributions, <u>not</u> the subtraction of expectation values in a "vacuum" or other state.

Existence of Time Ordered Products ("Renormalization")

Given the def. of Wick powers, the axioms themselves uniquely determine by induction (in the number of factors) the definition of

$$T\left(\phi^{k_1}(x_1)\cdots\phi^{k_n}(x_n)\right)$$

except on the "total diagonal" $x_1 = x_2 = \ldots = x_n$. The commutation condition implies a "local Wick expansion" in terms of $:\phi^{k_1-j_1}(x_1)\cdots\phi^{k_n-j_n}(x_n):_H$. Problem reduces to that of extending the c-number coeficients $t^0_{j_1\cdots j_n}(x_1,\ldots,x_n)$ to the total diagonal.

Key idea: We prove a "scaling expansion" for t^0 of the form,

$$t^0 = \sum_{k=0}^{m} \frac{1}{k!}\tau^0_k + r^0_m$$

where r^0_m has scaling behavior sufficient to guarantee that it can be uniquely extended to the total diagonal and each τ^0_k is of the form

$$\tau^0_k(x_1,\ldots,x_n) = C^{\mu_1\cdots\mu_k}(x_1)\, u^0(x_2,\ldots,x_n) \qquad (*)$$

where $C^{\mu_1\cdots\mu_k}$ is a local curvature expression involving a total of k derivatives of the metric, and u^0 corresponds to a Lorentz invariant distribution in the tangent space of x_1. (Eq. $(*)$ may be viewed as proving a generalized form of the "local momentum space" expansion assumed by Bunch & Parker.)

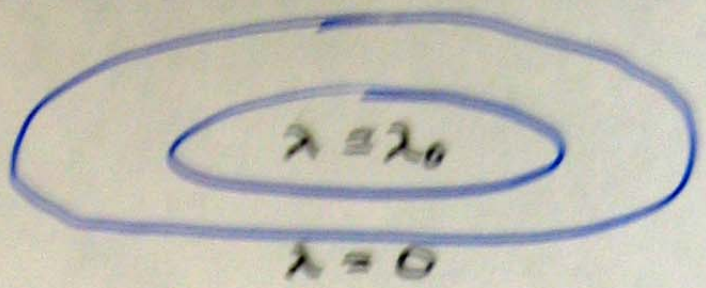
The distributions u^0 may then be extended to the origin (e.g. via the Minkowski spacetime methods of Epstein & Glazer) to yield a def. of time ordered products satisfying our axioms.

The Bogoliubov Formula

Consider

$$\mathcal{L} = \underbrace{\nabla_a \phi \nabla^a \phi + (\xi R + m^2)\phi^2}_{\mathcal{L}_0} + \underbrace{\lambda \phi^4}_{\mathcal{L}_1}$$

View λ as a smooth function of compact support:



so that action of $\mathcal{L}_1$ is $\phi^4(\lambda)$.

Define:

$$S[\mathcal{L}_1 = \lambda\phi^4] = \sum_n \frac{i^n}{n!} T(\underbrace{\phi^4(\lambda) \cdots \phi^4(\lambda)}_{n})$$

$$\boxed{\phi_I(x) = \frac{\delta}{i\delta f} S^{-1}[\mathcal{L}_1]\, S[\mathcal{L}_1 + f\phi]}$$

Bogoliubov formula for interacting field (to be interpreted as a formal power series expansion). Formula is adjusted so that $\phi_I = \phi$ before interaction is "turned on". Limit as $\lambda \to \lambda_0$ need not exist. However can modify formula to keep ϕ_I fixed "in the interior" of the spacetime & can then take limit as $\lambda \to \lambda_0$.

Upshot: The interacting field algebra for a quantum field in an arbitrary globally hyperbolic curved spacetime is well defined (as a formal power series).

Main Conclusions

For at least the case of a scalar field, free QFTCS has been given a mathematically precise formulation in which all Wick polynomials in the field are well defined (up to expected renormalization ambiguities). The effects of interactions can be calculated perturbatively and is rigorously defined to all orders and for arbitrary interactions.

This has been accomplished despite the absence of i) Poincare (or any other) symmetry, ii) a preferred "vacuum state" (or any other preferred state), iii) globally defined Fourier transforms, or iv) Euclideanization.

The resulting theory is entirely local and covariant in nature, and does not contain extraneous notions, such as "particles".