Boundary States and Modular Bootstrap in CFT

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§1. Boundary state & modular bootstrap

consider RCFT e.g. $SU(2)_k$ WZW

finite number of reps.

$$\ell = 0, 1, \ldots, k$$

character function

$$\chi_\ell(\tau) \equiv \mathrm{tr}_{R_\ell}\, q^{L_0 - \frac{c}{24}}, \quad q = e^{2\pi i \tau}$$

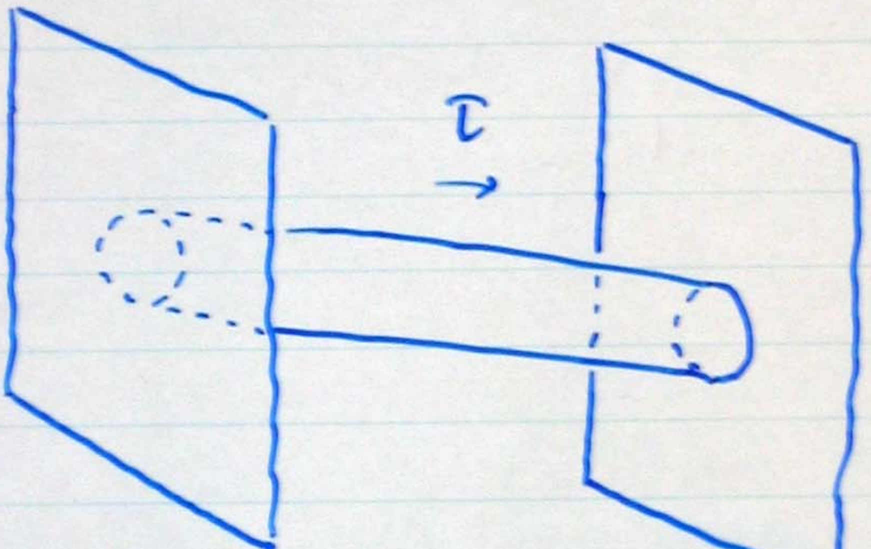
S transformation

$$\chi_\ell\left(-\frac{1}{\tau}\right) = \sum_{\ell'=0}^{k} S_{\ell\ell'}\, \chi_{\ell'}(\tau)$$

$$S_{\ell\ell'} = \sqrt{\frac{2}{k+2}}\,\sin\frac{\pi(\ell+1)(\ell'+1)}{k+2}$$

$\tau \to -\frac{1}{\tau}$: exchanging the role of 2 dim. space and time

Open-closed duality



boundary states

$(L_n - \tilde{L}_{-n})\,|B, \xi\rangle = 0$

$\langle B, \xi_1|$ $\qquad$ $|B, \xi_2\rangle$

D-branes

$$\langle B, \xi_1|\, e^{i\tau H^c}\, |B, \xi_2\rangle \qquad H^c = L_0 + \tilde{L}_0 - \frac{c}{12}$$

$$= \sum_{\xi} N_{\xi_1, \xi_2\, \xi}\, \chi_{\xi}\left(-\frac{1}{\tau}\right)$$

positive integer

Cardy

$|l\rangle\rangle$ Ishibashi states

$$\langle\langle l| e^{i\tau H^c} |l'\rangle\rangle = \delta_{ll'} \chi_l(\tau)$$

$$|B;L_1\rangle = \sum_l \frac{S_{lL_1}}{\sqrt{S_{l0}}} |l\rangle\rangle$$

$$\sum_l \frac{S_{lL_1} S_{lL_2}}{S_{l0}} \chi_l(\tau) = \sum_{l,L} \frac{S_{lL_1} S_{lL_2}}{S_{l0}} S_{lL} \chi_L(-\tfrac{1}{\tau})$$

$$= \sum_L N_{L_1 L_2 L} \chi_L(-\tfrac{1}{\tau})$$

Verlinde fusion coefficients

$$N_{0L_1L_2} = \delta_{L_1 L_2}$$

$$\therefore \langle B;0| e^{i\tau H^c} |B;L\rangle = \chi_L(-\tfrac{1}{\tau})$$

§ 2. Irrational CFT

What happens in the case of IRCFT?

Liouville theory

$$L = +\frac{1}{2}(\partial\phi)^2 + \mu e^{2b\phi} + QR\phi$$

$$Q = b + \frac{1}{b}$$

$$c = 1 + 6Q^2$$

$$h(e^{2\alpha\phi}) = \alpha(Q-\alpha) = -(\alpha-\frac{Q}{2})^2 + \frac{Q^2}{4}$$

$$= p^2 + \frac{Q^2}{4} \quad : \quad \alpha = ip + \frac{Q}{2}$$

$p \geq 0$ continuous rep.

$$= 0 \quad : \quad \alpha = Q$$

identity rep.

Use of Liouville theory

1. Non-critical string dual to Matrix Model
2. Critical string theory compactified on CY mfd. singular space-time.

$|B:0\rangle$ = ZZ - brane

(Zamolochikov^2)

$|B:p\rangle$ = FZZT - brane

(Fateev - Z^2 - Teschner)

spectrum of the theory

open $\begin{cases} \text{continuous} \\ \text{identity} \end{cases}$

closed continuous

$\Psi_0(p)$, $\Psi_p(p')$: boundary wave functions

agree with conformal bootstrap

§3. $N=2$ Liouville theory

T. E. & Y. Sugawara

For the application to string theory we consider $N=2$ Liouville.

$$(\phi, Y, \psi, \psi^*)$$

compact boson

$$c = 3(1+Q^2), \quad Q^2 \equiv \frac{K}{N}$$

$N=2$ Liouville is known to T-dual to $SL(2;\mathbb{R})/U(1)$ coset theory describing 2 dim. BH.

Unitary reps. $SL(2;\mathbb{R})/U(1)$

continuous rep. $j = \frac{1}{2} + \frac{ip}{Q}$

identity rep. $j = 0$

discrete rep. $1 \leq s \leq N+2K$, $j = \frac{s}{2K}$

$$\chi_p(-\tfrac{1}{\tau}) = \langle B,0|\, e^{i\tau H^c} \,|B;p\rangle$$

$$\chi_{n=0}(-\tfrac{1}{\tau}) = \langle B,0|\, e^{i\tau H^c} \,|B,0\rangle$$

$$\chi_p(\tau) = \frac{q^{p^2}}{\eta(\tau)}$$

modular transformation

$$\chi_p(\tfrac{-1}{\tau}) = \int_0^\infty dp' \cos(2\pi p p')\, \chi_{p'}(\tau)$$

$$\chi_{n=0}(\tfrac{-1}{\tau}) = \int_0^\infty dp' \sinh(2\pi b p') \sinh(\tfrac{2\pi p'}{b})\, \chi_{p'}(\tau)$$

$$\therefore \quad |B;0\rangle = \int_0^\infty dp\, \Psi_0(p)\, |p\rangle\rangle$$

where $|p\rangle\rangle$ is an Ishibashi state

$$\langle\langle p|\, e^{i\tau H^c} \,|p'\rangle\rangle = \delta(p-p')\, \chi_p(\tau)$$

$$\Psi_0(p) \sim \frac{2\pi i p}{\Gamma(1+2ipb)\,\Gamma(1+2i\frac{p}{b})}$$

$$|B:p\rangle = \int_0^\infty dp'\, \Psi_p(p')\, |p'\rangle\rangle$$

$$\Psi_p(p') \sim \frac{1}{p'}\, \Gamma(1-2ibp')\,\Gamma(1-2i\tfrac{p'}{b}) \cos 2\pi p p'$$

S transformation

continuous $\xrightarrow[S]{}$ continuous

discrete $\xrightarrow[S]{}$ continuous + discrete

identity $\xrightarrow[S]{}$ continuous + discrete

discrete rep. only in the range $K \leq S \leq N+K$

spectrum

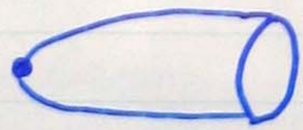
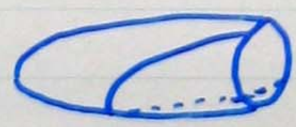
open $\left\{ \begin{array}{l} \text{continuous} \\ \text{discrete} \quad\quad 1 \leq S \leq N+2K \\ \text{identity} \end{array} \right.$

closed $\left\{ \begin{array}{l} \text{continuous} \\ \text{discrete} \quad\quad K \leq S \leq N+K \end{array} \right.$

$\updownarrow$

$SL(2:\mathbb{R})/U(1)$: $\frac{1}{2} \leq j \leq \frac{k+1}{2}$

These appear to be general features of IRCFT

Boundary states	$SL(2;\mathbb{R})/U(1)$
identity rep (A type)	D0 brane 
continuous rep (B type)	D1 
discrete rep (A type)	D2 
continuous rep (A type)	D2 

general models

$$\prod_i L_{(N_i, k_i)} \otimes \prod_j M_{k_j}$$

$N=2$ Liouville fields   minimal models

These describe

NS5

ALE spaces (fibered on $\mathbb{P}^1$)

singular CY_3, CY_4, ...

These are spaces of conifold type

deformed conifold	resolved conifold
complex str. def. only	Kähler def. only
$N=2$ Liouville	$SL(2;\mathbb{R})/U(1)$

Landau-Ginzburg description?

$N=2$ minimal models

$$W \sim g\,\Phi^{n} , \qquad n = k+2$$

computation of topological index

elliptic genus

$$\mathrm{tr}\,(-1)^{F_R+F_L}\, q^{L_0}\bar{q}^{\bar{L}_0} \times \exp 2\pi i J_0^L z$$

$$\sum_{l=0}^{k} ch^{\tilde{R}}_{l,l+1}(\tau; z) = \frac{\theta_1(\tau; (\frac{1}{n}-1)z)}{\theta_1(\tau; \frac{1}{n}z)}$$

Ramond ground states

chiral free field of

$U(1)_R$ charge $\frac{1}{n}$

Liouville sector

$$\sum_{s=K}^{N+K} ch^{\tilde{R}}_{dis\; s,s-K}(\tau; z) = \sum_m \frac{q^{NKm^2} e^{4\pi i K m}}{1 - q^m e^{\frac{2\pi i z}{N}}} \frac{\theta_1(\tau; z)}{\eta(\tau)}$$