1. Localization Theory

The fundamental localization concept suggested in the classical Anderson paper in 1958 is the subject of this introductory lecture. This is the very significant concept because it creates the language adequately describing elementary excitation in disordered system. I will introduce it assuming that the system is at zero temperature and there is one elementary excitation which we are going to study. The simplest model for describing this excitation is a tight-binding model. In such a model we have sites enumerated by the index $i=1, .. N$, which are coupled by transition amplitudes t_{ij} and have random energies ϕ_i characterized by the certain distribution function $f(\phi)$ centered at $\phi=0$ and having the width w. The Schrödinger equation for the elementary excitation can be written either in the time dependent or in the stationary form (Eqs. (1a,b) below)

$$i\hbar \frac{\partial c_i}{\partial t} = \phi_i c_i + \sum_k t_{ik} c_k \tag{1.1a}$$

$$E c_i = \phi_i c_i + \sum_k t_{ik} c_k . \tag{1.1b}$$

The second equation was obtained from the first one setting $c_i(t) = c_i e^{-iEt/\hbar}$. Disordering introduced in Eqs. (1.1a, b) is called diagonal disorder or site disorder. Alternatively, one can consider off-diagonal disorder, which is associated with the fluctuations of the transition amplitudes t_{ij}. It can be also characterized by some distribution function $g(t)$. The physical system leading to the model Eq. (1) is illustrated in Fig. 1.1. Site index corresponds to the lowest electronic energy level in the dopant potential, which is shown by the horizontal line. Diagonal disordering in local energies ϕ is caused by fluctuations in the shape of potential wells, while the off-diagonal disordering in transition (hopping) amplitudes t_{ij} can be due to fluctuations in distance between dopants.

Model Eqs. (1.1a,b) describes many systems. It was applied by P. W. Anderson in 1958 to study spin-waves in ferromagnets placed into random fluctuating magnetic field. This model also describes electrons in doped semiconductors. In this case disordering is caused by random positions of dopants as well as structural disorder and other impurities. It also describes phonons and excitons in irregular crystals and/or amorphous solids.

Anderson localization can be described as following. Depending on the energy E of elementary excitation Eq. (1.1b), characteristic disordering w and transition amplitudes t_{ij} the eigen state of the problem Eq. (1.1b) can be either localized (bound) or delocalized (unbound or extended). In the first case the wavefunction is essentially different from zero only in the finite domain of the space with the typical size l called the localization length. In the second case the wavefunction is essentially non-zero everywhere inside the sample. Formal definition of the localization can be given in terms of the so called participation ratio defined using the eigenstate wavefunction $\psi_E(x)$ at certain energy E

$$P_E = \int dx^d |\psi_E(x)|^4, \tag{1.2a}$$

In the case of a discrete problem Eq. (1.1b) the participation ration can be expressed as

$$P_E = \sum_{i=1}^{N} |c_i^E|^4 . \tag{1.2b}$$

Here d is the system dimension. It is straightforward to see that if the wavefunction is different from zero only in the restricted domain of the size l then the participation ratio can be estimated as

$$P \approx \frac{1}{l^d} \to l \approx P^{-1/d} \tag{1.3a}$$

which gives us a way to estimate the localization length (radius) in the continuous case. In a discrete case we have

$$P \approx \frac{1}{n} \to n \approx P^{-1} \tag{1.3b}$$

Where n is the number of sites occupied by the given localized state.

In the case of the wavefunction distributed uniformly in the whole sample of the size L the participation ratio is defined by the inverse sample volume. Thus the crucial difference between localization and delocalization regimes can be defined in terms of the behavior of the participation ratio in the limit of an infinite system size. When $L \to \infty$ the participation ratio remains finite for localized states and it approaches zero for delocalized states. For the discrete system the inverse participation ratio is equal to the number of sites occupied by the quasi-particle eigenstate.
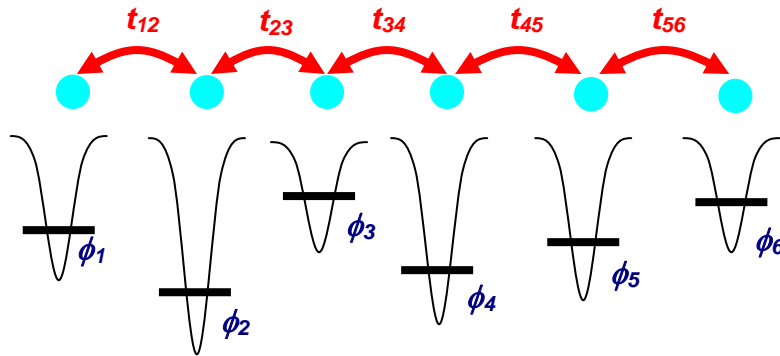

Fig. 1.1. Electron in disordered system (Eq. (1)).

In systems of high dimension $d>2$ the localization can be described using the hand-waving concept of resonances, which is quite useful for all considerations of this lecture set. I will introduce this concept using two sites following earlier work by Burin, Maksimov and Chesalov (1987) and Levitov (1990).

To study localization we start with the very simple system of one particle on two sites i, j (molecular orbitals, local vibrational excitations, magnetization deviation from its saturated value, etc.) coupled by transition amplitude t_{ij} and having local energies ϕ_i and ϕ_j. The Schrödinger equation Eq. (1.1b) for this system can be expressed in the form

$$Ec_i = \phi_i c_i + t_{ij} c_j, \quad Ec_i = \phi_i c_i + t_{ij} c_i. \tag{1.4}$$

The solutions to this equation can be expressed in the form of the standard u-v transform (e. g. Levitov, 1990)

$$\tilde{\psi}_i = \cos(\vartheta/2)|i> + \sin(\vartheta/2)|j>, \quad \tilde{\psi}_j = -\sin(\vartheta/2)|i> + \cos(\vartheta/2)|j>,$$

$$\tan(\vartheta) = \frac{2t_{ij}}{\phi_i - \phi_j}, \quad E_{i,j} = \frac{\phi_i + \phi_j}{2} \pm \sqrt{\left(\frac{\phi_i - \phi_j}{2}\right)^2 + t_{ij}^2} = \frac{\phi_i + \phi_j}{2} \pm \frac{\phi_i - \phi_j}{2\cos(\vartheta)}. \tag{1.5}$$

Wavefunctions $|i>$ and $|j>$ describe particles completely localized on corresponding sites. The extent of the wavefunction delocalization can be defined using participation ratio as

$$P_{i,j} = \cos^4(\vartheta/2) + \sin^4(\vartheta/2) = 1 - \frac{4t_{ij}^2}{2((\phi_i - \phi_j)^2 + 4t_{ij}^2)}. \tag{1.6}$$

The participation ratio reaches its minimum at precise resonance of two sites, when $|\phi_i - \phi_j| = 0$ and it is equal to ½ there, which means that the wavefunction is shared in equal proportions between two avail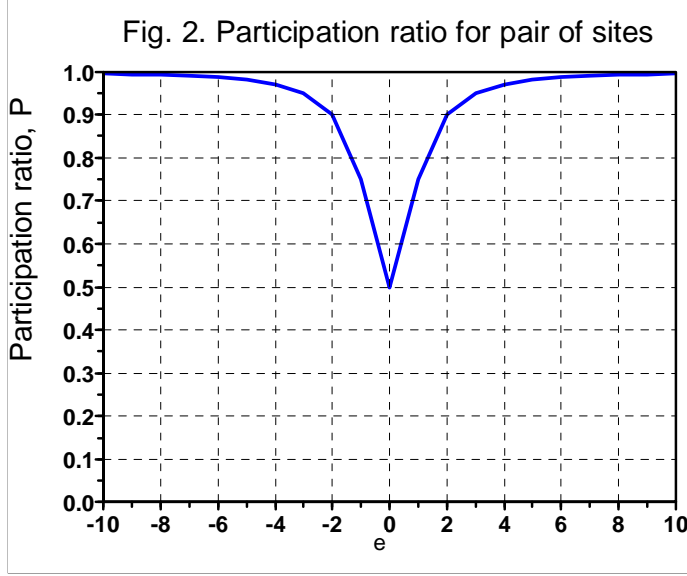able sites. Number of occupied sites is expressed precisely by the inverse participation ratio. This corresponds to the maximum possible extension of the wavefunction on two sites. In the opposite limit $|\phi_i - \phi_j| >> t_{ij}$ the participation ratio asymptotically approaches unity, which means that the eigenstate wavefunction is essentially localized on one of two sites.

Fig. 2. Participation ratio for pair of sites

One can clearly see in Fig. 2 where the participation ratio is shown as a function of the dimensionless ratio $e = |\phi_i - \phi_j|/t_{ij}$ that the wavefunction is essentially localized in one of the sites except for the case of **resonant** conditions which can be approximately expressed as

$$|\phi_i - \phi_j| < t_{ij}. \tag{1.7}$$

We can qualitatively define the pair of sites satisfying Eq. (1.7) as the resonant pair, while otherwise the pair of sites is non-resonant.

In the absence of the coupling between sites ($t_{ij}=0$) eigenstates of the problem (Eq. (1.4)) are localized on single sites and the participation ratio is equal to 1. Eq. (1.6) can be treated as the correction to the participation ratio in the fully localized states if we assume that this coupling is small. Since our system is random the participation ratio should be averaged over the distribution $f(x)$ of random energy difference $x = \phi_i - \phi_j$. This correction can be expressed as

$$\delta P_{i,j} = -\left\langle \frac{4t_{ij}^2}{2((\phi_i - \phi_j)^2 + 4t_{ij}^2)} \right\rangle = -\int_{-\infty}^{+\infty} \frac{f(x)4t_{ij}^2 dx}{2(x^2 + 4t_{ij}^2)}. \tag{1.8}$$

Assuming that the coupling of sites is small compared to the disordering one can approximate replace the function $f(x)$ by its value at $f(0)$ and then the straightforward integration yields

$$\delta P_{i,j} = -\pi f(0)|t_{ij}| \sim -\frac{|t_{ij}|}{W}. \tag{1.9}$$

It is interesting that the correction to the full localization participation ratio ($P=1$) is defined by the probability of resonance between sites i and j. Indeed the probability of

resonance Eq. (1.7) can be expressed as ratio of the size of energy domain for e. g. energy ϕ_i corresponding to the resonance, which is around $2t_{ij}$ and the size of total domain of energy, which is given by W. The integral in Eq. (1.8) is defined by the resonant domain Eq. (1.7). Thus we can conclude that the reduction in the participation ratio is associated with the formation of resonant pairs.

In the next step we extent the perturbation theory analysis to a more complicated situation where each site is coupled to many other sites. Then if the coupling strength is small compared to the characteristic disordering W one can first treat pairs of sites separately in the lowest order of perturbation theory. The corrections from all sites j Eq. (1.8) should be added to each other and we find

$$\delta P_i = -\sum_j \left\langle \frac{4t_{ij}^2}{2((\phi_i - \phi_j)^2 + 4t_{ij}^2)} \right\rangle \approx -\sum_j \frac{|t_{ij}|}{W}. \tag{1.10}$$

It is clear that depending on the value of the correction Eq. (1.10) compared to unity, which is the participation ratio in a fully localized state, one can either expect that the coupling t_{ij} is too weak to break down a localization so all eigenstates are localized. If the correction is much larger than unity one can expect that all states are delocalized. Indeed, the
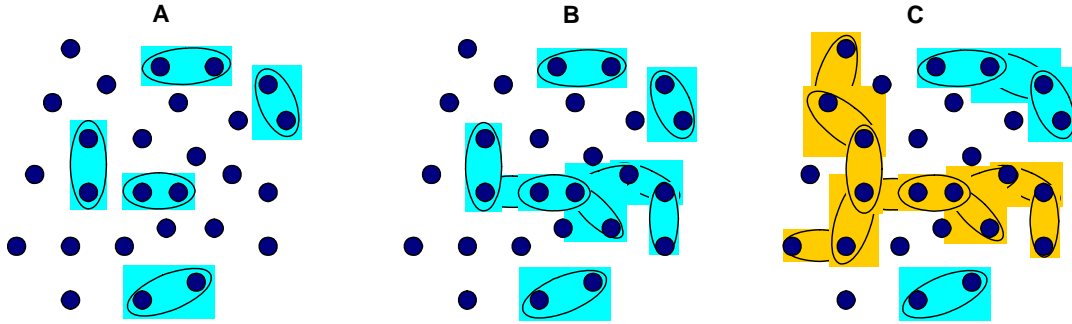

Fig. 1.3. Percolational approach to the delocalization problem. Resonant pairs are circled. In case A the number of resonances per site is much smaller than unity, in case B a number of resonances per site is of order of unity but still insufficient to form the infinite cluster, in case C the number of resonances exceeds one so the infinite cluster is formed (shown by yellow couplings))

correction Eq. (1.10) can be treated as the number of the resonant interaction per the given site i. If this number is much greater than one then one can create the infinite percolation cluster of coupled resonances (see Fig. 3) which promotes the delocalization of the quantum state through the whole sample.

The estimate of the number of resonances depends on the particular behavior of the transition amplitude t_{ij} with the distance. For electronic excitations it decreases with the distance according to the exponential law or faster (see Fig. 1) which can usually be replaced with the nearest neighbor interaction. Then we can assume that each site is coupled to Z other neighboring sites, while there is no coupling to sites located at longer distances. The sum in Eq. (1.10) can be expressed as

$$\delta P_{i,j} \approx -\frac{Zt_0}{W}. \tag{1.11}$$

where t_0 stands for the coupling at neighboring sites. This estimate permits us to get the localization criterion. If the disordering is strong

$$zt < W \tag{1.12}$$

then one can expect that eigenstates of the problem Eq. (1.1b) are localized while in the opposite limit they are delocalized.

Note that criterion Eq. (1.12) is derived for quantum states in the middle of the energy band with $E\sim0$. For eigenstates in the band tail $|E|>W$ Eq. (1.12) changes to

$$zt < \frac{1}{f(E)}. \qquad (1.13)$$

Since in the band tail $f(E)\to0$ the states at band edge are always localized.

Our consideration proves that if Eq. (1.11) is not satisfied then the perturbation theory starting with the strong localization of all states fails. Can we say what happens in the opposite limit? The alternative approach can be developed using the propagating waves as the basis and then considering corrections to some parameters of interest caused by disordering. One relevant approach to study the localization of quantum states with certain energy E is to evaluate the system conductivity assuming that we study the Fermi system at zero temperature and the energy of interest coincides with the Fermi energy. Then if the electronic states at the Fermi energy are localized the conductivity should be zero, while for delocalized states it is expected to be finite. We follow the work of Altshuller and coworkers. First we derive the Kubo formula for the conductivity and then use diagram technique to evaluate the conductivity.

Imagine that the system was in external alternating electric field with the frequency Ω that will be later set to zero. The field was turned on adiabatically slow starting at $t=-\infty$

$$\mathbf{E}(t) = \sum_{a=x,y,z}\mathbf{e}_a E_a \exp(-i\Omega t + \delta t). \qquad (1.14)$$

For a most general consideration we consider the system described by the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V}, \qquad (1.15)$$

where H_0 describes the system and V is the perturbation associated with the external field

$$\hat{V} = \exp(-i\Omega t + \delta t)\sum_{b=x,y,z}\hat{P}_b E_b, \qquad (1.16)$$

where P_b is the operator of the system dipole moment.

The conductivity is defined as the proportionality coefficient between the applied field and the induced current. The current can be found using the time-dependent density matrix calculated in the linear order in V. Using the Schrödinger equation for the density matrix we can evaluate it as

$$i\hbar\frac{\partial\rho}{\partial t} = [\hat{H},\rho] = [\hat{H}_0,\rho] + [\hat{V},\rho],$$

$$\rho \approx \exp\left(-\frac{i\hat{H}_0 t}{\hbar}\right)\rho_0\exp\left(\frac{i\hat{H}_0 t}{\hbar}\right) - \frac{i}{\hbar}\exp\left(-\frac{i\hat{H}_0 t}{\hbar}\right)\int_{-\infty}^t d\tau[\hat{V}(\tau),\rho_0]\exp\left(\frac{i\hat{H}_0 t}{\hbar}\right), \qquad (1.17)$$

$$\rho_0 \approx \exp\left(-\frac{\hat{H}_0 - F}{k_B T}\right).$$

Average current can be found using Eq. (1.17) as

$$< j_a >= Tr\{\rho \hat{j}_a\} = -\frac{i}{\hbar} \int_{-\infty}^{t} d\tau Tr\{[\hat{V}(\tau), \rho_0] \hat{j}_a(t)\} = -\frac{i}{\hbar} \int_{-\infty}^{t} d\tau Tr\{[\hat{j}_a(t), \hat{V}(\tau)] \rho_0\} =$$

$$= -\frac{i \exp(-i\Omega t)}{\hbar} \sum_b E_b \int_{-\infty}^{0} d\tau \langle \hat{j}_a(0), \hat{P}_b(\tau) \rangle \exp(-i\Omega\tau + \delta\tau). \tag{1.18}$$

Using Eq. (1.18) one can define the linear conductivity as

$$\sigma_{ab}(\Omega) = Tr\{\rho \hat{j}_a\} = -\frac{i}{\hbar} \int_{-\infty}^{0} d\tau \langle \hat{j}_a(0), \hat{P}_b(\tau) \rangle \exp(-i\Omega\tau + \delta\tau) =$$

$$= \frac{K_{ab}(0) - K_{ab}(\Omega + i\delta)}{V\hbar(\Omega + i\delta)}, \tag{1.19}$$

$$K_{ab}(\Omega) = \int_{0}^{+\infty} dt \langle \hat{j}_a(t), \hat{j}_b(0) \rangle \exp(i\Omega t - \delta t).$$

For the static conductivity one should take the limit of zero frequency.

Conductivity is the relevant characteristics for systems of charged particles. More general way applicable to delocalized excitations is to define the diffusion coefficient, which can be found using the Einstein relationship

$$D_{ab} = \frac{\sigma_{ab}}{e^2 dn / d\mu} = \frac{K_{ab}(0) - K_{ab}(\Omega + i\delta)}{Ve^2 dn / d\mu\hbar(\Omega + i\delta)}, \tag{1.20}$$

where $dn/d\mu$ is the derivative of the Fermi particle density over the chemical potential, taken at zero temperature. This equation can be derived from the equilibrium condition for the current through the system

$$\sigma \mathbf{E} - eD\nabla n = 0. \tag{1.21}$$

Considering the fact that $E = -\nabla\varphi$ and $n = n(\mu - e\varphi)$ and expanding the particle density over the small potential φ one can easily derive Eq. (1.20).

It is convenient to express all parameters using the system Green functions. In the noninteracting Fermi system one can express the Fermi particle density as

$$n = \frac{1}{V} \sum_{\alpha} \Theta(E - E_a) \tag{1.22}$$

where $E = \mu$ is the energy of the quantum state of interest. Then the derivative of the chemical potential reads

$$\frac{\partial n}{\partial \mu} = \frac{1}{V} \sum_{\alpha} \delta(E - E_a) = \frac{1}{V} \sum_i f_{ai} f_{ai}^* \delta(E - E_a)$$

$$= \frac{1}{2\pi i V} \sum_i \left[\frac{f_{ai} f_{ai}^*}{E - E_a - i\delta} - \frac{f_{ai} f_{ai}^*}{E - E_a + i\delta} \right], \tag{1.23}$$

where f_{ai} is the solution of the Schrödinger equation (1.1b) with the energy E_a

$$E_a f_{ia} = \phi_i f_{ia} + \sum_k t_{ik} f_{ka}. \tag{1.24}$$

and V is the system volume.

Eq. (1.23) can be rewritten in terms of retarded and advanced Green functions of the system as

$$\frac{\partial n}{\partial \mu} = \frac{N < G_{ii}^R(E) - G_{ii}^A(E) >}{2\pi i V}, \tag{1.25}$$

where average sign <...> stands for the configurational average over all possible realizations of disordering (potentials ϕ_i) and N is the number of lattice sites. The advantage of retarded and advanced Green functions is that they do not depend on the particle statistics and can be used for all quasi-particles of interest including electrons, phonons, photons, excitons and spin-waves. Below we also derive the expression for the conductivity of the non-interacting system Eq. (1.20) in terms of retarded and advanced Green functions. First we can define the operator of the dipole moment as

$$\hat{P}_a = -e\sum_i x_{ia} c_i^+ c_i. \tag{1.26}$$

Then the current operator can be defined as the time derivative of the dipole moment

$$\hat{j}_a = i[\hat{H}, \hat{P}_a] = -ie\sum_{i,j}(x_{ia} - x_{ja})t_{ij} c_i^+ c_j. \tag{1.27}$$

In the non-interacting system it is convenient to express the operator of current in terms of the basis of eigenstates Eq. (1.24)

$$\hat{j}_a(t) = -ie\sum_{i,j}(x_{ia} - x_{ja})t_{ij} f_{ia}^* f_{jb} c_a^+ c_b \exp\left(\frac{i(E_a - E_b)t}{\hbar}\right). \tag{1.28}$$

Then the real part of the current-current correlation function can be expressed as
$$[\hat{j}_a(t), \hat{j}_b(0)]_\Omega$$

$$= -\frac{\pi e^2}{\hbar^2}\sum_{i,j,k,l}(x_{ia} - x_{ja})(x_{kb} - x_{lb})t_{ij}t_{kl} f_{ia}^* f_{la} f_{kb}^* f_{jb}(n_a - n_b)\delta\left(\Omega + \frac{(E_a - E_b)}{\hbar}\right) =$$

$$= \frac{e^2}{4\pi\hbar^2}\sum_{i,j,k,l}(x_{ia} - x_{ja})(x_{kb} - x_{lb})t_{ij}t_{kl} \tag{1.29}$$

$$\times \int_{E-\hbar\Omega}^{E}\left(G_{kj}^R(E_1) - G_{kj}^A(E_1)\right)\left(G_{kj}^R(E_1 + \hbar\Omega) - G_{kj}^A(E_1 + \hbar\Omega)\right)dE_1$$

Substituting this expression into the definition of the conductivity Eq. (1.19) we got

$$\sigma_{ab} = \frac{e^2}{4\pi\hbar^2 V}\sum_{i,j,k,l}(x_{ia} - x_{ja})(x_{kb} - x_{lb})t_{ij}t_{kl} \times \left\langle\left(G_{kj}^R(E) - G_{kj}^A(E)\right)\left(G_{il}^R(E) - G_{il}^A(E)\right)\right\rangle. \tag{1.30}$$

For the system without off-diagonal disorder in the simplest approach of decoupling of the average Green function product this equation leads to the Drude formula. Indeed, we can represent it as

$$\sigma_{ab} = \frac{e^2 \delta_{ab}}{3\pi}\int\frac{d\mathbf{p}}{(2\pi\hbar)^d}\mathbf{v}_\mathbf{p}^2 \frac{\mathrm{Im}(\Sigma_E)^2}{\left[(E - \mathrm{Re}(\Sigma_E) - \varepsilon_\mathbf{p})^2 + \mathrm{Im}(\Sigma_E)^2\right]^2}, \tag{1.31}$$

where the Green function for the weakly scattered plane wave is used in the standard form

$$G_p^R(E) = \frac{1}{E - \Sigma_E - \varepsilon_\mathbf{p}}, \tag{1.32}$$

and the energy spectrum and group velocity of elementary excitations are defined as

$$\varepsilon_{\mathbf{p}} = \sum_j t_{ij} \exp\left(\frac{i\mathbf{p}\mathbf{r}_{ij}}{\hbar}\right), \quad \mathbf{v}_{\mathbf{p}} = \frac{\partial \varepsilon_{\mathbf{p}}}{\partial \mathbf{p}}. \tag{1.33}$$

v_0 stands for the volume of the elementary cell.

In the domain where the scattering is weak the integral in Eq. (1.31) can be approximately evaluated leading to the Drude expression

$$\sigma_{ab} = \frac{e^2 <v_F^2> \delta_{ab}}{3\pi \operatorname{Im}(\Sigma_E)} g_0(E_F) \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} = \frac{e^2 g_0(E) <v_F^2> \delta_{ab}}{6 \operatorname{Im}(\Sigma_E)} =$$

$$= \frac{e^2 g_0(E) <v_F^2> \tau \delta_{ab}}{3}, \tag{1.34}$$

where the density of states is defined as

$$g_0(E) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^d} \delta(E - E_{\mathbf{p}}), \tag{1.35}$$

the average squared velocity is defined as

$$<v_F^2> = \frac{1}{g_0(E)} \int \frac{d\mathbf{p}}{(2\pi\hbar)^d} \left(\frac{\partial \varepsilon_{\mathbf{p}}}{\partial \mathbf{p}}\right)^2 \delta(E - E_{\mathbf{p}}), \tag{1.36}$$

and the scattering time τ is given by

$$\tau = \frac{1}{2\operatorname{Im}\Sigma_E}. \tag{1.37}$$

The transition to the classical Drude formula can be made taking into account that the product $(2/3)g_0(E)(mv_F^2/2)$ is equal to the electron density n for the system with the spherical Fermi surface. This result in the Drude expression $\sigma = ne^2\tau/m$.

The approximation Eq. (1.33) is good when the scattering is weak. Below I discuss the effect of the strong scattering leading to the Anderson localization. The target of this study is to investigate corrections to the Drude formula Eq. (1.34) and see where they become critically important. We will show that they are significant under conditions when the corrections to the single site localization are negligible. Since the problem of diagonal disorder has been investigated by many people and for our future study the off-diagonal disorder is more important we consider the model Eq. (1.1b) with the strong off-diagonal disorder assuming that coupling constants t_{ij} are significantly different from zero for $z>>1$ neighbors of each site. We also assume that

$$<t_{ij}> = 0 \tag{1.38}$$

and it is characterized by some distribution function $h(t_{ij})$ which is symmetric with respect to 0. This model is relevant to the case where the transitions of excitation are associated with the dipolar or elastic interactions like for excitons or spin-waves as well as for electrons in the strong magnetic field fluctuating in space. In the absence of diagonal disorder the model is close to the Wigner random matrix and we will reproduce several answers known for that system.

To characterize the system conductivity we need to calculate its configurational averaged Green functions. We start with the single particle retarded Green function satisfying the following equation

$$E^+ g_{i0}^R = \delta_{i0} + \phi_i g_{i0}^R + \sum_k t_{ik} g_{k0}^R, \quad E^+ = E + i\delta. \tag{1.39}$$

Here g stands for the Green function not averaged over configurations (distribution of energies). The result should be averaged during its calculation. The configurationally averaged Green function will be denoted by the capital letter G so

$$G_{i0}^{R} = < g_{i0}^{R} >. \tag{1.40}$$

We will use the perturbation theory method expanding the Green function in the leading order in the large parameter $z >> 1$. The same result can be obtained using the Σ-model but I prefer an iterative approach as most straightforward and qualitatively clear one.

Eq. (1.39) can be rewritten as

$$g_{i0}^{R} = \frac{\delta_{i0}}{E^{+} - \phi_{i}} + \frac{1}{E^{+} - \phi_{i}} \sum_{k} t_{ik} g_{k0}^{R}. \tag{1.41}$$

The first term in the right hand side of Eq. (1.41) can serve as the zero approximation while the second term is the correction. Then we can substitute it to the second term. Repeating this procedure an infinite number of times we can rewrite the solution for the Green function as the infinite series

$$g_{i0}^{R} = \frac{\delta_{i0}}{E^{+} - \phi_{i}} + \frac{t_{i0}}{\left(E^{+} - \phi_{i}\right)\left(E^{+} - \phi_{0}\right)} + \sum_{k} \frac{t_{ik} t_{k0}}{\left(E^{+} - \phi_{i}\right)\left(E^{+} - \phi_{k}\right)\left(E^{+} - \phi_{0}\right)} +$$

$$+ \sum_{k,l} \frac{t_{ik} t_{kl} t_{l0}}{\left(E^{+} - \phi_{i}\right)\left(E^{+} - \phi_{k}\right)\left(E^{+} - \phi_{l}\right)\left(E^{+} - \phi_{0}\right)} + ... \tag{1.42}$$

After configurational average of the series the only diagonal G_{00} Green function differs from zero because all transition terms t_{kl} should be met twice in Eq. (1.42) so that the path over sites must lead to exactly the same point, where it has started. Thus
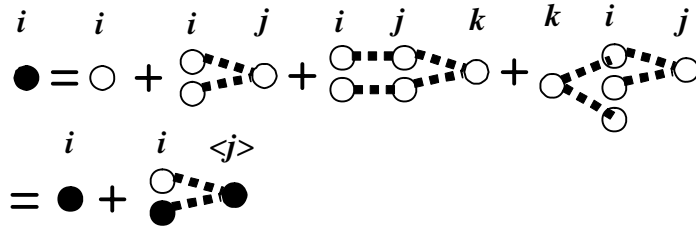

Fig. 1.4. Diagram approach to the single particle Green function. Open circle stands for the zero point Green functions and dashed line for transition matrix elements t_{ij}.

$$G_{i0} = G\delta_{i0}. \tag{1.43}$$

This property serves as an excellent illustration of the well recognized fact that the configurationally averaged single particle Green function contains no direct information about the localization transition in the system. Moreover the Ioffe-Regel criterion telling us that the localization takes place when the mean free path of excitations gets comparable to their wavelength is fully irrelevant for the problem we study because elementary excitations cannot be represented as plane waves in the model with strong off-diagonal disorder. Despite of all those circumstances it is evident since the Wigner's consideration that in the case of a large number of neighbors z representative excitations of the problem are delocalized.

Consider the terms in Eq. (1.42) non-vanishing after configurational average. First terms in the leading order in z are given by

$$G^R = \int_{-\infty}^{+\infty} \frac{P(\phi_i)d\phi_i}{E^+ - \phi_i} + \sum_k \int_{-\infty}^{+\infty} \frac{P(\phi_i)d\phi_i}{\left(E^+ - \phi_i\right)^2} \int_{-\infty}^{+\infty} \frac{<t_{ik}^2> P(\phi_k)d\phi_k}{\left(E^+ - \phi_k\right)}$$

$$+ \sum_{k,l} \int_{-\infty}^{+\infty} \frac{P(\phi_i)d\phi_i}{\left(E^+ - \phi_i\right)^3} \int_{-\infty}^{+\infty} \frac{<t_{ik}^2> P(\phi_k)d\phi_k}{\left(E^+ - \phi_k\right)} \int_{-\infty}^{+\infty} \frac{<t_{il}^2> P(\phi_l)d\phi_l}{\left(E^+ - \phi_l\right)} \qquad (1.44)$$

$$+ \sum_{k,l} \int_{-\infty}^{+\infty} \frac{P(\phi_i)d\phi_i}{\left(E^+ - \phi_i\right)^2} \int_{-\infty}^{+\infty} \frac{<t_{ik}^2> P(\phi_k)d\phi_k}{\left(E^+ - \phi_k\right)^2} \int_{-\infty}^{+\infty} \frac{<t_{kl}^2> P(\phi_l)d\phi_l}{\left(E^+ - \phi_l\right)}.$$

Neglected terms are related to the case where the third index k in last two terms in Eq. (1.44) coincides with i. This term is definitely small as $1/z$ compared to other terms in the series.

It is convenient to represent our expansion using diagrams shown in Fig. 4. In the leading order in z one can express the iteration series self-consistently as

$$G_i = \frac{1}{E^+ - \phi_i}\left(1 + zt_0^2 GG_i\right), \qquad (1.45)$$

where G_i is the Green function at site i averaged over all disordering except for the random potential ϕ_i. Eq. (1.45) is algebraic and we can solve it finding the closed equation for the Green function G in the form

$$G = \int_{-\infty}^{+\infty} \frac{f(\phi)d\phi}{E^+ - \phi - zt^2 G}. \qquad (1.46)$$

In the limit of weak disordering

$$f(E)^2 zt^2 << 1 \qquad (1.47)$$

one can set $\phi=0$ in Eq. (1.45) or replace $f(\phi)=\delta(\phi)$ in Eq. (1.46). Then the Green function takes the well known semicircular form which expresses the density of states for the eigenmodes of random matrix

$$G^{R,A} = \frac{E}{2zt^2} \pm \frac{i}{2zt^2} \sqrt{4zt^2 - E^2}, E < 2t\sqrt{z},$$

$$G^{R,A} = \frac{E \pm i\delta}{2zt^2} - \frac{E \pm i\delta}{2zt^2} \sqrt{1 - \frac{4zt^2}{(E \pm i\delta)^2}}, E > 2t\sqrt{z}. \qquad (1.48)$$

The density of states is defined by the imaginary part of this Green function. It differs from 0 for $|E| < 2t\sqrt{z}$ or otherwise there is no eigenstates in the remaining energy domain.

The important value to study is the density of states which can be expressed through the imaginary part of the Green function G. Taking imaginary part of both sides of Eq. (1.46) we got

$$\text{Im}\, G = zt^2 \,\text{Im}\, G \int_{-\infty}^{+\infty} \frac{f(\phi)d\phi}{|E^+ - \phi - zt^2\,\text{Re}\, G|^2 + zt^2(\text{Im}\, G)^2}$$

$$\rightarrow 1 = zt^2 \int_{-\infty}^{+\infty} \frac{f(\phi)d\phi}{|E^+ - \phi - zt^2\,\text{Re}\, G|^2 + zt^2(\text{Im}\, G)^2}. \qquad (1.49)$$

Eq. (1.49) is the important identity which is the Word identity in the leading order in z. In the limit of weak disordering it leads to the Green function Eq. (1.48) while in the limit of

strong disordering the imaginary part of the Green function is defined by the distribution function f as

$$\operatorname{Im} G \approx \pi f(E). \tag{1.50}$$

In this limit corrections to the density of states obtained within the single site approach $G \approx 1/(E-\phi)$ are negligible.

Consider the two particle Green function, which is relevant for our study of particle transport. The propagation of excitations can be adequately described by the average product of two Green functions (retarded and advanced) taken at similar coordinates and slightly different energies

$$K_{ij}^{E,\Omega} = \left\langle \operatorname{Im} G_{ij}^{R}(E+\Omega/2) \operatorname{Im} G_{ij}^{A}(E-\Omega/2) \right\rangle. \tag{1.51}$$

This function is the partial contribution of quantum states with energy E to the Fourier expansion of the density – density correlation function $<\delta n_j(t)\delta n_i(0)>$ with the frequency Ω, which describes the propagation of particle density. One can define the backwards Fourier transform of the function Eq. (1.51) as

$$K_{ij}^{E}(t) = \int\limits_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} K_{ij}^{E,\Omega}. \tag{1.52}$$

The limiting behavior of this function taken at $i=j$ at $t \to \infty$ in the infinite sample characterizes whether the information originally stored at the position i. If the correlation function becomes 0 it means that the quantum states are delocalized so the particle gets infinitely far from its original location in the infinite time limit. If it is finite then the quantum states are localized. Below we show following Gorkov and Berezinskii that $K_{ii}(\infty)$ is proportional to the average participation ratio of the quantum states with energy E. One can represent the correlation function Eq. (1.51) as

$$K_{ii}^{E,\Omega} = \pi^2 \sum_{a,b} f_i^a * f_i^a \delta\left(E+\frac{\Omega}{2}-E_a\right) f_i^b * f_i^b \delta\left(E+\frac{\Omega}{2}-E_b\right). \tag{1.53}$$

The Fourier transform leads to the expression

$$K_{ii}^{E}(t) = \pi^2 \sum_{a,b} f_i^a * f_i^a f_i^b * f_i^b \delta(2E-E_a-E_b) \exp\left(-\frac{i(E_a-E_b)t}{\hbar}\right). \tag{1.54}$$

In the limit of an infinite time all terms disappears due to oscillations except for those with $a=b$. Then the expression of interest takes the form

$$K_{ii}^{E}(\infty) = \pi^2 \sum_{a} f_i^a * f_i^a f_i^a * f_i^a \delta(E_a-E). \tag{1.55}$$

Configurational averaging of this expression permits us to rewrite it in the form of averaged participation ratio at the given energy E as

$$\left\langle K_{ii}^{E}(\infty) \right\rangle = \frac{\pi^2}{N} \sum_{a} P_a \delta(E_a-E). \tag{1.56}$$

Fig. 1.5. Diagram approach to the two particle Green function. Notations are similar to Fig. 4.

Earlier we demonstrated that the participation ratio remains finite and close to unity for strong disordering $zt<W$ Eq. (1.12). Now

we study the correlation function Eq. (1.51) in the opposite limit expanding it over large parameter z as we did for the single particle Green function. Diagram series leading in z is shown in Fig. 5. It is most convenient to evaluate the whole series using the Fourier transform with respect to coordinate j. Then the series becomes the geometric series, which can be evaluated as

$$K_\Omega^E = \frac{\int\limits_{-\infty}^{+\infty} f(\phi)d\phi\left(\dfrac{1}{E^+ +\Omega/2-\phi-zt^2 G_{E+\Omega/2}^E} \times \dfrac{1}{E^- -\Omega/2-\phi-zt^2 G_{E-\Omega/2}^A}\right)}{1-\left(zt^2\right)_\mathbf{q}\int\limits_{-\infty}^{+\infty} f(\phi)d\phi\left(\dfrac{1}{E^+ +\Omega/2-\phi-zt^2 G_{E+\Omega/2}^E} \times \dfrac{1}{E^- -\Omega/2-\phi-zt^2 G_{E-\Omega/2}^A}\right)} \quad (1.57)$$

$$\left(zt^2\right)_\mathbf{q} = \sum_j <t_{ij}^2> \exp(i\mathbf{q}\mathbf{r}_{ij}).$$

Using the identity Eq. (1.49) and considering long time and distance limit corresponding to $\Omega, q \to 0$ we get

$$K_{\Omega,\mathbf{q}}^E \approx \frac{1}{zt^2}\frac{1}{1-\dfrac{\left(zt^2\right)_\mathbf{q}}{zt^2}+\dfrac{i\Omega}{2zt^2\,\mathrm{Im}\,G}} = \frac{2\,\mathrm{Im}\,G}{i\Omega+Dq^2}, D = \frac{zt^2\,\mathrm{Im}\,G}{d}\frac{\sum\limits_j t_{ij}^2 r_{ij}^2}{zt^2}. \quad (1.58)$$

In the limit of strong disordering Eq. (1.47) one can express the diffusion coefficient as

$$D = \frac{2zt^2}{d}\pi g(E)a^2 \approx \frac{2zt^2}{Wd}\pi a^2, a^2 = \frac{\sum\limits_j t_{ij}^2 r_{ij}^2}{zt^2}. \quad (1.59)$$

According to our findings Eqs. (1.58), (1.59), one can approximate the dynamics of quantum particle by the classical diffusion. This is definitely not fully true because the classical diffusion description does not contain interference which can be critically important for the quantum system. As we will see below the interference expressed by the *1/z* corrections lead to localization of certain quantum states in 3-d systems. Moreover all states in *1-d* and *2-d* systems so the quantum phenomena can be really important. Nevertheless above the localization transition the classical diffusion description is very good for the *3-d* quantum system.

To verify the applicability of this result for the diffusion coefficient we consider the Kubo expression for conductivity Eq. (1.30) in the leading order in z and then will study corrections to it. Actually, the order of correction in z corresponds to the number of loops in diagram expansion (like Figs. 1.4, 1.5 has no loops but each additional visit to the site creates the loop). The simplest approach for the conductivity Eq. (1.30) having no loops is defined by decoupling of average product of two Green functions

$$\sigma^{(0)} = \frac{e^2}{4\pi\hbar^2 V}\sum_{i,j,k,l}(x_{ia}-x_{ja})(x_{kb}-x_{lb})t_{ij}t_{kl}\left\langle\left(G_{kj}^R(E)-G_{kj}^A(E)\right)\right\rangle\left\langle\left(G_{il}^R(E)-G_{il}^A(E)\right)\right\rangle, \quad (1.60)$$
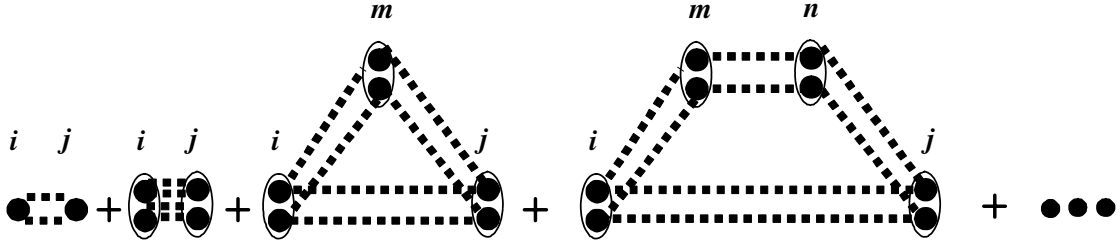
Fig. 1.6. Diagrams for the conductivity in leading order in z and the correction to it taken assuming $i{=}k$, $j{=}l$ in Eq. (1.60). All single loop approach corrections are shown. Factors x_{ij} are

described by the first diagram in Fig. 6. Since configurationally averaged Green functions are diagonal in coordinates one can re-express the zero order conductivity as

$$\sigma^{(0)} = \frac{e^2 (\mathrm{Im}\,G)^2 N}{\pi \hbar V d} \sum_j t_{ij}^2 x_{ij}^2. \tag{1.61}$$

Diffusion coefficient can be found using Einstein relationship Eq. (1.20) and since $dn/d\mu = N\mathrm{Im}G/(\pi V)$ we got the result

$$D^{(0)} = \frac{(\mathrm{Im}\,G)}{d} \sum_j t_{ij}^2 x_{ij}^2 = \frac{z t^2 a^2 (\mathrm{Im}\,G)}{d}, \tag{1.62}$$

which coincides with the earlier result Eq. (1.59). This coincidence confirms the consistency of our approach.

Consider corrections to the diffusion coefficient describing the localization transition. The first order correction in $1/z$ is defined by single loop diagrams. It is also important that only average products of retarded and advanced Green functions give the significant contribution to the configurationally averaged results because such a product has poles in energy complex plane on different sides with respect to the real axis, while if we have similar Green functions the poles are on the same side with respect to the real axis. The critical difference in both integrals is illustrated below assuming the Lorentzian distribution of the random potential $f(\phi) = \dfrac{W}{\pi (W^2 + \phi^2)}$ (although this can be proved for any distribution) assuming the validity of Eq. (1.47) that the density of states is defined by the random potential

$$< G^R G^A > = \int_{-\infty}^{+\infty} \frac{f(\phi)d\phi}{|\, E - \phi - z t^2 \,\mathrm{Re}\,G\,|^2 + z t^2 (\mathrm{Im}\,G)^2} \approx \frac{1}{W\sqrt{z t^2}\,\mathrm{Im}\,G},$$

$$< G^R G^R > = \int_{-\infty}^{+\infty} \frac{f(\phi)d\phi}{(E - \phi - z t^2 G^R)^2} \approx \frac{1}{(E - iW - z t^2 G^R)^2} \sim \frac{1}{W^2}, \tag{1.63}$$

$$< G^A G^A > = \int_{-\infty}^{+\infty} \frac{f(\phi)d\phi}{(E - \phi - z t^2 G^A)^2} \approx \frac{1}{(E - iW - z t^2 G^A)^2} \sim \frac{1}{W^2}.$$

In the case when disordering determines the density of states Eq. (1.47) (where as we will see below the corrections to the diffusion coefficient can be interesting) one can clearly neglect second and third integrals compared to the first one. It is also important that

because of the same reasons the only $<G^R G^A>$ correlation function has the diffusion pole while it does not exist for the products of identical Green functions.

In Fig. 6 the simplest sequence of single loop diagrams is shown. Although there more possible choices for the single loop series only this one possesses the proper sequence of products $<G^R G^A>$. Also for our estimate purposes it is sufficient to see on the example of typical correction in Fig. 6 where the perturbation theory in z becomes irrelevant first.

Since the sum over m, n and other intermediate sites in Fig. 1.6 can be reduced to the geometric series Fig. 1.5, Eq. (1.57), we can use here these results to express the correction to the conductivity Eq. (1.61) as

$$\delta\sigma = -\frac{e^2}{4\pi\hbar^2 Vd}\sum_{i,j} x_{ij}^2 t_{ij}^2 \left\langle G_{ij}^R(E)G_{ij}^A(E)\right\rangle = -\frac{e^2 N}{\pi\hbar^2 Vd}\sum_j \frac{x_{ij}^2 t_{ij}^2}{zt^2 a^2}\left\langle\frac{1}{q^2}\right\rangle_{ij}. \qquad (1.64)$$

It is important that the sign of the correction to conductivity is negative because of the product $x_{ij} x_{ji} = -x_{ij}^2$ in Eq. (1.30). The result is crucially sensitive to the system dimension because of the infrared divergence of inverse Fourier transform of $1/q^2$ for $d=1, 2$. We start with the consideration of the 3-d case and then will discuss lower dimension.

In the case d=3 one can represent the inverse Fourier transform of $1/q^2$ as

$$\left\langle\frac{1}{q^2}\right\rangle_{ij} = \frac{V}{4\pi r_{ij} N}. \qquad (1.65)$$

Then the correction (1.64) can be evaluated as

$$\delta\sigma = -\frac{e^2}{4\pi^2\hbar 3}\sum_j \frac{r_{ij} t_{ij}^2}{zt^2 a^2} \sim -\frac{e^2}{4a\pi^2\hbar 3}. \qquad (1.66)$$

This result should be compared to the zero point approximation Eq. (1.61) to estimate its applicability. Taking the ratio of the correction to the zero order conductivity we get

$$\frac{\delta\sigma}{\sigma} \sim -\frac{W^2 V}{Nzt^2 a^3}. \qquad (1.67)$$

Since the volume a^3 where a is the effective radius of hopping exceeds the elementary cell volume V/N by the large number of neighbors factor z one get

$$\frac{\delta\sigma}{\sigma} \sim -\frac{W^2}{z^2 t^2}. \qquad (1.68)$$

Thus the corrections to the conductivity become important in the localization regime defined by Eq. (1.12). Our consideration from both sides gives similar estimate of the localization criterion. There is no way to improve our estimate using the perturbation theory because all corrections from either side in the crossover regime $zt\sim W$ are comparable to each other and numerical there is required there. To my knowledge no analytical simplifications can be made near the transition point. Numerical study confirms the qualitative behavior described above.

Consider the case of the low dimension. For $d=2$ one has instead of Eq. (1.65)

$$\left\langle\frac{1}{q^2}\right\rangle_{ij} = \frac{V}{2\pi N}\ln\left(\frac{L}{r_{ij}}\right), \qquad (1.69)$$

where L is the system size. We will discuss the physical nature of this size dependence later after calculating the correction. This correction can be expressed as

$$\delta\sigma \sim -\frac{e^2}{4\pi^2\hbar}\ln\left(\frac{L}{a}\right).$$

(1.70)

In a one-dimensional case one can express the Fourier transform as

$$\left\langle\frac{1}{q^2}\right\rangle_{ij} = \frac{L}{\pi},$$

(1.71)

and we get

$$\delta\sigma \sim -\frac{e^2}{4\pi^2\hbar}L.$$

(1.72)

Thus in a *3-d* case our estimates confirms the naïve picture of percolational delocalization (see Fig. 1), while in low-dimensional systems the corrections to the conductivity depend on the systems size and become infinite when the system size approaching infinity. This reflects the full localization of all quantum states in model Eq. (1) in low dimension d=1, 2. The correction has the form of the number of returns of the diffusion path to its origin during the infinite time. As we know the probability to return back in one and two dimensional systems is unity after sufficiently long random walk while it is finite in a *3-d* case. Quantum mechanical return back is coherent in sense that the particle continues "following the loop" that it made initially.

Localization radius of quantum states in low dimension can be estimated setting the correction to the conductivity to its zero point value. This radius behaves as

$$\ln L \sim \left(\frac{zt}{W}\right)^2$$

(1.73)

which emphasizes the exponential sensitivity of localization to disordering. At weak disordering this radius is very large and can easily reach macroscopically large value so than the localization can be ignored. At finite temperature the maximum radius in Eq. (1.69) must be replaced by some radius where phase coherence fails and since this radius increases with decreasing temperature the conductivity has corrections logarithmically dependent on temperature. These effects predicted by Altshuller, Aronov, Larkin, and Khmelnitzkii, are called the weak localization and have been seen in many experiments.

It is interesting that qualitatively similar behavior of corrections to conductivity was found in scaling theory of localization by Abrahams, Anderson, Licciardello and Ramakrishnan. The only assumption they made is that when they consider the composed cubic sample of size *2L* made of *2^d* cubic samples of the size *L* the conductance of the big sample is the function of conductances of smaller samples. This hypothesis can be expressed using scaling equation

$$\frac{d\ln\Sigma}{d\ln L} = f(\Sigma).$$

(1.74)

In the case of a dimension *d* one would expect that for very large conductance Σ the system behaves classically meaning *f(Σ)≈d-2* because the conductance of the classical system behaves as *L^{d-2}*. The corrections due to quantum effects are expected to be negative and increasing with decreasing conductance so one can assume the expansion over *1/Σ* like

$$f(\Sigma) = d - 2 - \frac{A}{\Sigma}. \qquad (1.75)$$

In three dimensional systems this lead to the weak correction while in the case of d=2 Eq. (1.74) takes the form

$$\frac{d \ln \Sigma}{d \ln L} = -\frac{A}{\Sigma}. \qquad (1.76)$$

Accordingly its solution has the logarithmic correction dependent on the system size

$$\Sigma = \Sigma_0 - A \ln\left(\frac{L}{L_0}\right). \qquad (1.77)$$

This agrees with calculations expressed by Eq. (1.70) and moreover the constant A has a universal value of order of quantum conductance $e^2/\hbar$.

In a *1-d* case we got similarly

$$\sigma = \sigma_0 - A(L - L_0),$$
$$\sigma = L\Sigma. \qquad (1.78)$$

This equation also reproduces the quantum correction and in all cases the value of the coefficient A is close to the quantum conductance $e^2/\hbar$.

Localized states can be usually characterized by the localization length l which is defined by the exponential decrease of the wavefunction far from the center of the quantum state

$$\psi(r) \sim \exp(-r/l). \qquad (1.79)$$

The consideration should be made differently if the system Hamiltonian is not invariant with respect to the time-reversal symmetry. In the limit of strong random magnetic field we should assume the random phase on each hopping amplitude t_{ij} which leads to the following correlation rules

$$< t_{ij} > = 0, \; < t_{ij}^2 > = 0, \; < t_{ij} t_{ji} > = 0. \quad (1.80)$$

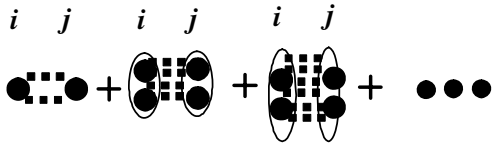$$i \quad j \quad\quad i \quad j \quad\quad i \quad j$$

Fig. 1.7. Diagrams for the conductivity in case of the strong random magnetic field

In this case all single loop terms in Fig. 6 except for the first one give zero contribution to the diffusion coefficient. Correction associated with the first term and other similar terms (Fig. 1.7) can be interpreted as the effect of the level repulsion on the particle capability to leave the site *i* for the site *j*. Namely the repulsion of energy levels by the transfer matrix element t_{ij} leads to the difficulty to conserve energy during the particle hopping. The conservation of energy is required with the accuracy of the width of energy level which can be estimated using the diffusion coefficient as

$$\frac{\hbar}{\tau} = \frac{zt^2}{W}. \qquad (1.81)$$

When the level repulsion (1.81) exceeds the hopping amplitude *t* one can ignore repulsion effects and this can be done at *zt>W*. This result coincides with earlier obtained criterion of delocalization. In the opposite limit the level repulsion cannot be neglected and one can expect particle localization.

It is possible to include the level repulsion effect self-consistently into the equation for the width of the energy level Eq. (1.81) for estimate purposes only. The result (1.81) can be reexpressed in the form of the Born approximation

$$\Gamma = \frac{\hbar}{\tau} = 2\pi \sum_j |t_{ij}^2| < \delta(E - \phi_j) >, \tag{1.82}$$

where E is the energy (ϕ_i) of the site i. In reality this δ-function should be replaced with some function with the width which is defined by the same self-consistent parameter (1.82)

$$\Gamma = \frac{\hbar}{\tau} = \frac{2\pi}{\Gamma} \sum_j |t_{ij}^2| F\left(\frac{E - \phi_j}{\Gamma}\right),$$

$$\int_{-\infty}^{+\infty} F(x)dx = 1. \tag{1.83}$$

We assume that the function F(x) has a maximum at x=0 with the width of order of unity.

Finally level repulsion can be included in the very straightforward manner as

$$\Gamma = \frac{\hbar}{\tau} = \frac{2\pi}{\Gamma} \sum_j |t_{ij}^2| \left\langle F\left(\frac{\sqrt{(E - \phi_j)^2 + |t_{ij}|^2}}{\Gamma}\right)\right\rangle_j. \tag{1.84}$$

Averaging leads to the replacement of function F/Γ with some dimensionless function G of the ratio t_{ij}/Γ having maximum at $t=0$.

$$\Gamma = \frac{\hbar}{\tau} = \frac{2}{W} \sum_j |t_{ij}^2| G\left(\frac{t_{ij}}{\Gamma}\right),$$

$$G(0) = 1.$$ $$\tag{1.85}$$

Since the t_{ij} is not equal zero for Z neighbors and takes the same value there one can rewrite Eq. (1.85) as

$$\Gamma = \frac{\hbar}{\tau} = \frac{2zt^2}{W} G\left(\frac{t}{\Gamma}\right). \tag{1.86}$$

If $zt>W$ the ratio in the argument of the function G is much less then unity so we can replace this function by unity and then we reproduce the earlier answer. If zt<<W, then the arguments is much greater than 1. Even if an analytical expansion for the function G at large argument works e. g. we can set it in the form G(x)=c/x², c~1 (the function G is expected to be even because both signs of Γ can be used in Eq. (1.84)) then we get

$$\Gamma = c\frac{2z\Gamma^2}{W}. \tag{1.87}$$

This equation has two solutions $\Gamma=0$ and $\Gamma \sim W/z$ and we should choose zero because second choice conflicts with our assumption $\Gamma<t$.

This method gives the right estimate for the localization transition point from the upper side but it cannot be used to characterize the transition because there are other effects which were not considered, including e. g. corrections in Fig. 1.6. It is not applicable therefore to low-dimensional systems.

References

Burin, Maksimov and Chesalov (1987) A. L. Burin, L. A. Maksimov, M. M. Chesalov, Propagation and Localization of spin excitations in uniaxial ferromagnets, Fizika Tverdogo Tela (Russian Journal of Solid State Physics), v. 29, pp. 3410-3420, 1987.

Levitov (1989) Europhysics Letters.

Levitov (1990) L. S. Levitov, Phys. Rev. Lett. **64**, 547 (1990);

Gorkov and Berezinskii, localization criterion.

Altshuller, Aronov, … 1979

Abrahams, Anderson, Licciardello and Ramakrishnan

Some sample problems

1. Derive the Kubo formula for the thermal conductivity (Zubarev, 1970)
2. Derive analytical expressions for the conductivity and diffusion coefficient using the simplest approach in the Wigner model without diagonal disorder and in the Lloyd model where the distribution of the random potential is given by the Lorentzian distribution $P(\phi) = \dfrac{W}{\pi\left(W^2 + \phi^2\right)}$.
3. Assume that the system described by Eq. (1) is in the strong random magnetic field so that $<t_{ij}t_{ji}>=t^2$, while $<t_{ij}^2>=0$. Investigate what happens with the correction to the conductivity in that case in 2-d system. Does it have logarithmic increase with the system size? How to handle the situation in the scaling localization theory.
4. Derive Eq. (2.4) considering four point charges $+q_1$ at position $\boldsymbol{a_1}/2$, $-q_1$ at position $-\boldsymbol{a_1}/2$, $+q_2$ at position $r+\boldsymbol{a_2}/2$, $-q_2$ at position $r-\boldsymbol{a_2}/2$ assuming that a_1, $a_2 << r$ and expanding the Coulomb interaction of couples of charges belonging to different dipoles over small lengths a and b. Remember that $\boldsymbol{d_1}=q_1\boldsymbol{a_1}$, $\boldsymbol{d_2}=q_2\boldsymbol{a_2}$.