

2. Localization and long-range coupling.

We are getting to the topic more closely related to the target of this class. We consider the problem of localization in case of essentially non-local coupling t_{ij} in Eq. (1.1). The mathematical example of the long-range coupling is the power law distance dependence

$$t_{ij} = \frac{t_0(\mathbf{\Omega})}{r_{ij}^a}, \quad (2.1)$$

where t_0 is the interaction constant possibly dependent on angles and other parameters (i. e. transition dipole moments of local states for energy transfer problem, etc.), but having no essential dependence on the distance r_{ij} .

What is the physical source of this power law coupling? One can consider the example of excitation transferred between two atoms or molecules having nonzero transition matrix elements of the dipolar interaction. Then the dipole moment operator of each molecule contains the part

$$\hat{\mathbf{d}} = \mathbf{d}_0(c^+ + c), \quad (2.2)$$

and \mathbf{d}_0 is the value of the transition dipole moment while operator c^+ creates the excited state and the operator c annihilates it. If two molecules are located closer than the resonant wavelength of light

$$r_{ij} < \lambda = \frac{2\pi c}{\omega} \quad (2.3)$$

Then the interaction of them takes the standard form

$$\hat{V}_{12} = \frac{\hat{\mathbf{d}}_1 \hat{\mathbf{d}}_2 - 3(\mathbf{n} \hat{\mathbf{d}}_1)(\mathbf{n} \hat{\mathbf{d}}_2)}{r_{ij}^3}, \quad \mathbf{n} = \frac{\mathbf{r}_{ij}}{r_{ij}}. \quad (2.4)$$

Interaction in the form Eq. (2.4) leads to the excitation transfer Hamiltonian in the resonant approach

$$\hat{V}_{12} = \frac{\mathbf{d}_1 \mathbf{d}_2 - 3(\mathbf{n} \mathbf{d}_1)(\mathbf{n} \mathbf{d}_2)}{r_{ij}^3} (c_1^+ c_2 + c_2^+ c_1). \quad (2.5)$$

Similarly elastic interaction in solids leads to the $1/r^3$ - long-range energy transfer. If dipolar moments are zero perhaps because of the symmetry then there exist quadrupole interaction decreases with the distance still by the power law $1/r^5$. It seems that the long-range amplitude of excitation transfer is unavoidable except for the mass transfer (electrons or heavy particles). The interaction radius is restricted by the wavelength of the quanta responsible for excitation virtual exchange which can be estimated as $\lambda_T \sim v\hbar/kT$, where v is the excitation velocity that is the speed of light for the dipolar interaction, the speed of sound for the elastic interaction or the Fermi velocity for the indirect exchange interaction. At low temperature this length is very large. For instance at $T=1K$ the thermal wavelength of light is of order of 1cm which is comparable to the typical size of the experimental sample so we can ignore the corresponding constraint.

Consider what happens if we are using the transfer amplitude Eq. (2.1), assume that excitations hops between centers placed randomly in the space with the density n and each center is in some random potential ϕ characterized by the distribution $f(\phi)$. Also we assume that the angular average transition amplitude is 0, which is certainly true for the multipole interaction. The problem is similar to the one considered previously for the short-range interaction except for the interaction distance dependence.

First of all we restrict the consideration to the case of not very small exponent a in Eq. (2.1). We assume $a > d/2$, otherwise all characteristic rate expressions like $\sum_j t_{ij}^2$ diverge at long distances so it is not clear how to treat such a system. I do not know any physical system with hopping amplitude characterized by the small exponent $a < d/2$.

Next we assume that disorder is strong in the sense of Eq. (1.47), so that the typical value of the sum of squared interactions $\sum_j t_{ij}^2 \sim V_0^2 n^{2a/d}$ is less than disordering in energy

$$V_0 n^{a/d} < W \sim 1/f(0). \quad (2.6)$$

Consider the correction to the participation ratio Eq. (1.10) in the case of the power law interaction t_{ij} Eq. (2.1)

$$\delta P_i = -\sum_j \left\langle \frac{t_{ij}^2}{2((\phi_i - \phi_j)^2 + t_{ij}^2)} \right\rangle = -\frac{n}{2} \int dr \int_{-\infty}^{+\infty} d\phi f(\phi) \left\langle \frac{t_0^2 / r^{2a}}{2((\phi_i - \phi_j)^2 + t_0^2 / r^{2a})} \right\rangle \quad (2.7)$$

where averaging is made with respect to the interaction amplitude. Changing the order of integration and evaluating integral over coordinate r we get

$$\delta P_i = -F \left(\frac{2a}{2a-d} \right) v_d n \frac{\langle |U_0|^{d/a} \rangle}{2} \int_{-\infty}^{+\infty} \frac{d\phi f(\phi)}{|\phi|^{d/a}}, \quad (2.8)$$

$$F(k) = \int_0^{+\infty} \frac{dx}{1+x^k} = \frac{\pi}{k \sin\left(\frac{\pi}{k}\right)}.$$

where v_d is the volume of a d -dimensional unit sphere,

The correction diverges at $a \leq d$. This is because the number of resonances diverges for the infinite volume of the system. The divergency in the number of resonances is caused by the divergence of the sum of absolute values of t_{ij} at $a \leq d$ at long distances. Since at small transition amplitude t_{ij} the probability of resonance is proportional to $|t_{ij}|$, the total number of resonances given by the sum of such probabilities diverges correspondingly. The delocalization becomes significant starting with the certain size R where the number of resonances becomes large. Restricting the sum over resonances in Eq. (2.7) by the certain value of $R=R_{\max}$ one can find

$$N_{res}(R_{\max}) \approx n f(0) t_0 R_{\max}^{d-a}, \quad a < d \quad (2.9)$$

$$N_{res}(R_{\max}) \approx n f(0) t_0 \ln(R_{\max} n^{1/d}), \quad a = d.$$

Correspondingly at $a < d$ one can expect that the delocalization shows up at the distance

$$R_* \approx n^{-1/d} \frac{1}{(f(0) t_0 n^{a/d})^{d/(d-a)}}, \quad a < d; \quad (2.10)$$

$$R_* \approx n^{-1/d} \exp\left(\frac{\eta}{f(0) t_0 n}\right), \quad a = d.$$

The first estimate is indeed relevant while as we will see below the second estimate does not really give us the delocalization criterion because the logarithmic divergence is too weak. For $a < d$ one can then estimate the rate of excitation hopping out of the site as

$$r \approx \frac{t_0}{R_*^a \hbar} = t_0 n^{a/d} \left(f(0) U n^{a/d} \right)^{\frac{a}{d-a}}, \quad (2.11)$$

which resolves the delocalization problem.

In the case of $a > d$ the excitations are localized when Eq. (2.6) is satisfied and the long-range interaction cannot assist them.

Can we describe the situation for $d/2 < a \leq d$ using the delocalization regime? One can apply the earlier suggested self-consistent approximation Eq. (1.84) to study the average transition rate of excitation from the given site

$$\Gamma = \frac{\hbar}{\tau} = \frac{2\pi}{\Gamma} \sum_j |t_{ij}^2| \left\langle F \left(\frac{\sqrt{(E - \phi_j)^2 + |t_{ij}|^2}}{\Gamma} \right) \right\rangle_j. \quad (2.12)$$

After averaging over random potential we obtain the result similar to Eq. (1.85)

$$\Gamma = \frac{\hbar}{\tau} = \frac{2}{W} \sum_j |t_{ij}^2| G \left(\frac{t_{ij}}{\Gamma} \right), \quad (2.13)$$

$$G(0) = 1.$$

The sum in the right hand side of Eq. (2.13) can be evaluated as

$$\Gamma = \frac{2nt_0^2}{W} \int \frac{d\mathbf{r}}{r^{2a}} G \left(\frac{t_0}{\Gamma r^a} \right) = \eta \left(\frac{a}{2a-d} \right) v_d \frac{2nt_0^2}{W} \left(\frac{\Gamma}{U_0} \right)^{\frac{2a-d}{a}}, \quad (2.14)$$

$$\eta(y) = \int_0^{+\infty} dx G(x^y) \sim 1,$$

where v_d is the volume of the d -dimensional sphere of the unit radius.

Eq. (2.14) definitely has the nonzero solution for the case of strong disordering only for $a < d$. This solution can be expressed as

$$\Gamma = C_a t_0 n^{a/d} \left(\frac{U n^{a/d}}{W} \right)^{\frac{a}{d-a}}, \quad (2.14.a)$$

This result agrees with our previous estimate Eq. (2.11) and with our percolational qualitative picture Fig. 1.3.

The difficult but important situation is related to the threshold case $a=d$. This is most interesting and most often met case of the long-range transfer amplitude. Eq. (2.14) has only trivial solution in this case despite of the divergency in the number of resonances in the infinite volume limit. For certainty we consider a $3-d$ case with a $1/r^3$ interaction. In this case it follows from the perturbation theory expansion Eqs. (2.7), (2.8) that all states must be delocalized. On the other hand our estimate of the excitation transfer rate Eq. (2.11) yields to the answer zero in the regime of strong disordering.

This problem has been resolved by Levitov in 1990, who pointed out that the delocalization characterized by the zero diffusion coefficient takes place so the situation is intermediate between the localization and strong delocalization. His arguments essentially used the concept of resonance and our discussion below is based on those arguments.

The special feature of the delocalization problem in the case of the $1/r^3$ is logarithmic divergence of the number of resonances with the system size. Eq. (2.13) in this case takes the form

$$N(R) = \frac{nv_d U_0}{W} \int_0^R \frac{dr^3}{r^3} = \frac{n4\pi t_0}{W} \ln\left(n^{\frac{1}{3}} R\right). \quad (2.15)$$

Accordingly one can estimate the distance to the k -th resonance ($k=1, 2, 3, \dots$) as

$$R_k = n^{-\frac{1}{3}} \exp\left(\frac{Wk}{4\pi m t_0}\right). \quad (2.16)$$

Thus the distance to each next resonance increases by approximately the same factor compared to the previous one. This factor is very large and we cannot interpret the situation using the percolational scenario.

Indeed, one can describe the evolution of system wavefunctions with the increase of the interaction radius following Levitov 1990. In each step the interaction radius increases by some factor η ,

$$1 \ll \eta \ll \exp\left(\frac{W}{4\pi m U_0}\right). \quad (2.15)$$

Imaging that interaction radius increases in the step k from R_k to $R_{k+1}=R_k\eta$. Also imagine that in the step k all quantum states of the system are localized and have localization radius much smaller than R_k . In the step k the small fraction of the quantum states, namely

$$\chi = \frac{n4\pi U_0}{W} \ln(\eta) \ll 1 \quad (2.16)$$

will have resonant interaction with some other states because they started feeling each

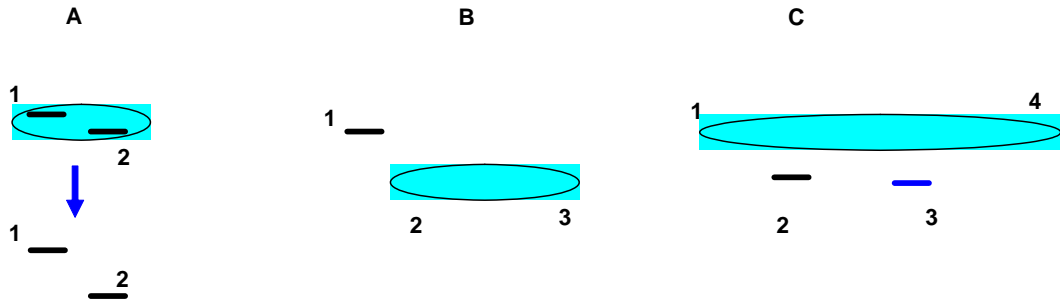


Fig. 2.1. Resonant interactions of quantum states in case of $1/R^3$ interaction and stepwise increase of interaction radius from case A to case C as described in text. At step 1 resonance takes place between states 1 and 2 which results in their level repulsion. Because of that in the step 2 only the modified state 2 got to resonance with the state 3, while in the step 3 only the modified state 1 got in resonance with the state 4 which prevent the formation of the percolation cluster

other through the long-range interaction. Note that if two states were in resonance before at smaller radius then only one of two newly formed superpositions will participate in the next resonance. This is because the level repulsion given by the resonant interaction in the previous step is much larger than the resonant interaction in the present step (see Fig. 2.1).

The probability of the single resonance in each given step is small Eq. (2.16). The probability of two resonances is given by even smaller parameter χ^2 (χ^n for n resonances

simultaneously). Therefore there is no way to form infinite percolation cluster from resonances in each step of the consideration. For each resonance event localization radius of the particular quantum state increases and reaches the interaction radius. However since next resonance takes place at the interaction radius bigger by the factor $\exp\left(\frac{W}{4\pi m U_0}\right)$ than the one for the previous resonance accordingly localization radius

remains much smaller in average than the interaction radius. Therefore we can treat quantum states as point-like states during all consideration. Collective states are formed though because each state has infinite number of resonances with increasing the interaction radius. It was demonstrated by Levitov in 1990 that for such a system coupling constants for collective states have a quite reasonable nearly Gaussian statistics.

So what can we say about the quantum states in a strongly disordered system with $1/r^3$ amplitude of excitation transfer? They are definitely delocalized because the number of resonances increases unlimitedly with increasing the interaction radius and each resonance reduces the participation ratio of the given quantum state by some factor η greater than 1. On the other hand the participation ratio of the finite system of the size L will scale as

$$P(L) \propto \eta^{-\chi \ln(L/a)} \propto \left(\frac{a}{L}\right)^{-d'}, \quad d' = \chi \ln \eta \quad (2.17)$$

For typical delocalized states occupying all available volume one would expect d' to be equal to the system dimension 3. This is not the case for our system where $d' < 1$. So the delocalized states are fractal (actually multifractal) with very small effective dimensionality at strong disordering. These delocalized states behave similarly to the threshold states in the short-range interaction problem with the energy separating the localized states from delocalized. It is interesting that dynamics is also similar in those two systems.

The diffusion coefficient can be estimated using the following arguments. Using the uncertainty principle for time and energy one can consider the increase of interaction radius as the time evolution of the system with the characteristic time

$$t \sim \frac{\hbar R^3}{U_0}. \quad (2.18)$$

Since localization radius increases proportionally to the interaction radius and the localization radius tells us where the excitation has been transferred one can obtain the following relationship

$$r(t) \sim t^{1/3}. \quad (2.19)$$

Remember that for the normal diffusion we have $r(t) \sim t^{1/2}$. Thus we have the anomalously slow diffusion regime with the diffusion coefficient scaling with the size as

$$D(L) \propto \frac{1}{L}. \quad (2.20)$$

This behavior is identical to the diffusion coefficient behavior at the localization threshold as follows from the scaling theory of localization. If we take Eq. (1.74) for a 3-d system we should set the function f to be 0 at the threshold. Then the system

conductance is constant which corresponds to conductivity and diffusion coefficient behaving as in Eq. (2.20).

Note that the logarithmic increase of resonance number Eq. (2.15) is very slow and therefore if the logarithmic prefactor is really small (for instance, it is less than 0.01 for interacting two-level systems in glasses) then delocalization can be ignored for any realistic system size. Despite of that delocalization at finite temperature is still possible even there because of the many-body interaction which we are going to consider later.

Literature

L. S. Levitov, Phys. Rev. Lett. 1990.