

## 5. Two Level Systems in Glasses. Interaction Stimulated Delocalization of energy.

In this lecture we approach the problem of relaxation and decoherence caused by TLS interaction. In contrast with the previous lecture we will consider it using the approach based on Anderson localization theory. The interaction with any thermal bath will not be included into this consideration. We will also discuss the consequences of the energy delocalization for other system of interest.

The Hamiltonian of the model of interacting TLS can be written as

$$\hat{H} = \hat{H}_0 + \hat{V}_{\text{int}},$$

$$\hat{H}_0 = -\sum_i (\Delta_i S_i^z + \Delta_{0i} S_i^x) \quad (5.1)$$

$$\hat{V}_{\text{int}} = \frac{1}{2} \sum_{i \neq j} U_{ij} S_i^z S_j^z.$$

Here  $U_{ij}$  is  $1/R^3$  interaction of TLS induced by either their elastic or dipolar interactions. For our purposes it can be expressed in the general form

$$|U_{ij}| \sim \frac{U_0}{R_{ij}^3}, \quad (5.2)$$

where  $U_0$  is the characteristic interaction constant defined as

$$U_0 \approx \frac{\mu^2}{\varepsilon}, \quad (5.3)$$

where  $\mu$  is a TLS dipole moment and  $\varepsilon$  is the dielectric constant of the glass or

$$U_0 \approx \frac{\gamma^2}{\rho v^2}, \quad (5.4)$$

where  $\gamma$  is a TLS – phonon interaction constant,  $\rho$  is the density of glass and  $v$  is the velocity of sound.

Most important parameter of theory is the dimensionless product  $P_0 U_0$  which has a universal and small value in all known glasses

$$P_0 U_0 \sim 10^{-3}. \quad (5.5)$$

Our problem of interest is whether some typical TLS  $i$  with tunneling amplitude  $\Delta_0$  and energy  $E$  ( $E < kT$ ) being initially in excited state will be able to transfer its energy irreversibly to other TLS and if yes, then how long would it take?

The delocalization takes place due to resonant interactions. Each resonant interaction can include this TLS  $i$  and any other TLS sequence  $i_1, i_2, i_3, \dots, i_n$ . Their simultaneous transition can be characterized by the energy difference of initial and final states

$$\Delta_n = -E + \sum_{k=1}^n E_k \Delta S_k^z, \quad (5.6)$$

and collective transition amplitude

$$V_n \quad (5.7)$$

which depends on TLS parameters and interactions in a very complex manner. Under resonance conditions one has

$$|\Delta_n| \ll |V_n|. \quad (5.8)$$

Each coupling  $V_n$  is related to some energy scale  $\varepsilon \sim V_n$  that characterizes the rate of transition through this resonance channel. In our considerations we will start with large

energies of order of the thermal energy  $kT$  and then reduce the energy scale of consideration verifying in each step whether the interaction is strong (the number of resonances is greater or equal one which assumes the energy delocalization) or it is weak and no resonances take place. Until the interaction is weak we can remove the corresponding energy scale from the consideration as adiabatically fast with the proper modification of parameters of remaining low energy excitations. Since for weak interaction the modification is weak we can use this analysis as the estimate of the transition point if any exists. It is, for instance, perfectly applicable to the Anderson localization problem for the system with coupling strength decreasing with the distance as  $1/R^a$  ( $a \leq 3$ ).

The simplest transition leading to sharing TLS energy is the transition involving two TLS including the one under consideration ( $E, \Delta_0$ ) and some other TLS  $i_1$  separated from the given one by the distance  $R$  and having energy  $E_1$  and tunneling amplitude  $\Delta_{01}$ . The energy change associated with the transition can be expressed as

$$\Delta_p = |E - E_1| \quad (5.9)$$

while the transition amplitude takes the form in the second order of perturbation theory Eq. (4.19)

$$V_p = \frac{U_0}{R^3} \frac{\Delta_0}{E} \frac{\Delta_{01}}{E_1}. \quad (5.10)$$

The resonant condition can be written as

$$|E - E_1| < \frac{U_0}{R^3} \frac{\Delta_0}{E} \frac{\Delta_{01}}{E_1}. \quad (5.11)$$

The number of resonances in the energy domain  $\varepsilon/2 < V_p < \varepsilon$  can be estimated as

$$\begin{aligned} N_s(\varepsilon) &\approx P_0 \int d^3R \int_0^{+\infty} dE_1 \int_0^{E_1} \frac{d\Delta_{01}}{\Delta_{01}} \theta\left(\frac{U_0}{R^3} \frac{\Delta_0}{E} \frac{\Delta_{01}}{E_1} - |E - E_1|\right) \\ &\times \theta\left(\frac{U_0}{R^3} \frac{\Delta_0}{E} \frac{\Delta_{01}}{E_1} - \frac{\varepsilon}{2}\right) \theta\left(\varepsilon - \frac{U_0}{R^3} \frac{\Delta_0}{E} \frac{\Delta_{01}}{E_1}\right) \approx P_0 U_0 \frac{\Delta_0}{E} \sim 10^{-3} \frac{\Delta_0}{E} \ll 1 \end{aligned} \quad (5.12)$$

It is clear from the above estimate that the probability to find resonance in the TLS-TLS interaction channel is negligibly small due to the weakness of interaction Eq. (5.5). Therefore more complicated channels should be added into consideration.

Next channel is made of this excited TLS plus two others. As previously we consider the TLS coupled to two neighbors and characterized some arbitrary parameters  $E, \Delta_0 < kT$ . Two others have energies and tunneling amplitude  $E_1, \Delta_{01}$  and  $E_2, \Delta_{02}$ . If both of them are initially in their ground states their contribution to the resonant situation is of order of  $N_s^2$  Eq. (5.12) because the situation is essentially equivalent to the Anderson localization problem. If one of them is in its ground state ( $S_1$ ), while the other one is in its excited state ( $S_2$ ) then the qualitatively new effect compared to the single particle Anderson localization is originated from the collective transition including all three spins ( $S_1, S_2, S_3 \rightarrow -S_1, -S_2, -S_3$ ), which cannot take place in the single particle problem. The main phase volume for such neighboring spins is associated with the situation when both TLS are the thermal ones (see Eq. (4.7))

$$\Delta_0 \sim E \sim kT. \quad (5.13)$$

Also the most significant contribution to the resonances is due to the situation when these two thermal TLS are located closer to each other than the main one. Their interaction should be stronger because their energies are bigger. Thus the case of interest is defined by the given TLS in its excited state and two thermal TLS, one excited and one in its ground state. Two thermal TLS are separated from each other by the distance  $R_1$ , while the given TLS is separated from them by the distance  $R$ .

The pair of thermal TLS can be treated as a two-level system with respect to the flip-flop transition involving both of them. This flip flop transition can be characterized by energies and tunneling amplitudes (cf. Eqs. (5.9), (5.10))

$$\begin{aligned} \Delta_{0p} &= \frac{U_0}{R_1^3}, \\ \Delta_p &= E_1 - E_2. \end{aligned} \quad (5.14)$$

Accordingly the energy and transition amplitude for three TLS simultaneously can be expressed as

$$V_t = \frac{U_0}{R^3} \frac{\Delta_0}{E} \frac{\Delta_{op}}{\sqrt{\Delta_p^2 + \Delta_{0p}^2}}. \quad (5.15)$$

The number of resonances at characteristic energy  $\varepsilon$  can be expressed as

$$\begin{aligned} N_t(\varepsilon) &= P_0^2 T \int d^3 R \int d^3 R_1 \\ &\times \int_{-T}^T d\Delta_p \theta(V_t - |\Delta_p - E|) \theta\left(V_t - \frac{\varepsilon}{2}\right) \theta(\varepsilon - V_t) \theta(R - R_1) = \\ &= \frac{(P_0 U_0)^2 T \Delta_0}{E^2} \int d^3 R \int d^3 R_1 \theta\left(\frac{U_0^2 \Delta_0}{R^3 R_1^3 E^2} - \frac{\varepsilon}{2}\right) \theta\left(\varepsilon - \frac{U_0^2 \Delta_0}{R^3 R_1^3 E^2}\right) \\ &\times \theta\left(E - \frac{U_0}{R_1^3}\right) \frac{1}{R^3 R_1^3} = \frac{(P_0 U_0)^2 T \Delta_0}{E^2} \ln\left(\frac{E}{\varepsilon}\right) \end{aligned} \quad (5.16)$$

This is very interesting result. Indeed it shows that two level systems with energy  $E \sim \sqrt{(P_0 U_0)^2 T \Delta_0}$ .

Particularly for resonant TLS with  $E \sim \Delta_0 < E_* = T(P_0 U_0)^2$  the number of resonant interactions always exceeds 1. Remember that the corresponding frequency  $E_*/\hbar$  coincides with the previously estimated rate of spectral diffusion Eq. (4.28).

This calculation can be redone for the case of d-dimensional space and  $U_0/R^a$  interaction. In that case we get

$$\begin{aligned}
N_{ta}(\varepsilon) &= P_0^2 T \int d^d R \int d^d R_1 \\
&\times \int_{-T}^T d\Delta_p \theta \left( \frac{\Delta_0}{E} \frac{U_0^2}{R^a R_1^a \sqrt{\Delta_p^2 + \frac{U_0^2}{R_1^{2a}}}} - |\Delta_p - E| \right) \theta(R - R_1) = \\
&= \frac{(P_0 U_0)^2 T \Delta_0}{E^2} \int d^d R \int d^d R_1 \theta \left( E - \frac{U_0}{R_1^a} \right) \frac{\theta(R - R_1)}{R^a R_1^a} = \\
&= \frac{(P_0 U_0)^2 T \Delta_0}{E^2} \left( \frac{E}{U_0} \right)^{\frac{2(a-d)}{a}} \sim \frac{\Delta_0}{E} E^{-\frac{(2d-a)}{a}}.
\end{aligned} \tag{5.18}$$

Similarly to the previous lecture we get the strong effect at small energies for the interaction decreasing with the distance slower than  $1/R^{2d}$ . For resonant TLS with  $E \sim \Delta_0$  the number of resonant interaction is always greater than one if the TLS energy is below its critical value

$$E_* = W_T \left( \frac{W_T}{T} \right)^{\frac{d}{2d-a}}, \tag{5.19}$$

$$W_T = U_0 (P_0 T)^{\frac{a}{d}}.$$

Why did we get what we got? This can be understood if we consider rare resonant pairs interacting with each other. For instance, for  $1/R^3$  interaction in 3-d space the probability of resonance for two thermal TLS in any energy domain ( $\varepsilon/2 < \Delta_0 p < \varepsilon$ ) or spatial domain ( $U_0/\varepsilon < R^3 < 2U_0/\varepsilon$ ) is small but finite Eq. (5.12). It is constant  $P_0 U_0$ , independent both of the TLS interaction energy and the characteristic distance between them. The density of resonant pairs is thus constant for any energy scale and it is given by the product of the density of thermal TLS in excited state  $P_0 T$  and the probability that another thermal TLS is in resonance with the given one which is  $P_0 U_0$  in the given energy domain, so resonant pair density is given by

$$n_{res} = P_0 T P_0 U_0. \tag{5.20}$$

There exists interaction between resonant pairs capable to transfer energy between them. It is also  $U_0/R^3$  interaction of the entirely many-body nature. Since all pairs are in resonance the interaction has the same form as the original interaction of z-projections of pseudospins. The typical coupling strength is given by the interaction of adjacent pairs which is the product of interaction constant and average (5.20) expressing the inverse cubed distance between neighbors

$$V_* \sim T (P_0 U_0)^2. \tag{5.21}$$

Note that the interaction Eq. (5.21) does not depend on pair energies so for the typical pair subsystem energy  $\varepsilon < V_*$  we arrive at the situation when the coupling of pairs exceeds their characteristic energy disordering. This is the situation for Anderson delocalization transition. Reduction of typical energy  $\varepsilon$  can make the Anderson delocalization parameter (number of resonances) arbitrarily large so the delocalization transition is inevitable. We considered off-diagonal interaction. Diagonal interaction cannot break down

delocalization because it's induced disordering scales with the number of neighbors as  $z^{1/2}$  while the delocalization effect is proportional to the number of neighbors  $z$ . Our analysis explains the nature of the large number of resonances Eq. (5.16) for TLS energy less than  $E_*$ . Similarly one can treat the situation for the interaction  $U_0/R^a$  in a  $d$ -dimensional space.

The probability to form a resonant pair with the energy of order of  $\varepsilon$  for the given excited thermal TLS is given by

$$P_p^a(\varepsilon) \approx P_0 \int d^d R \int_0^{+\infty} dE_1 \int_0^{E_1} \frac{d\Delta_{01}}{\Delta_{01}} \theta\left(\frac{U_0}{R^a} - |E - E_1|\right) \times \theta\left(\frac{U_0}{R^a} - \frac{\varepsilon}{2}\right) \theta\left(\varepsilon - \frac{U_0}{R^a}\right) \approx P_0 U_0 \left(\frac{\varepsilon}{U_0}\right)^{\frac{a-d}{a}}. \quad (5.22)$$

The density of resonant pair of the certain energy  $\varepsilon$  is given by the product of the density of thermal pairs and the resonance probability (5.22)

$$n_{res}^a = P_0 T P_0 U_0 \left(\frac{\varepsilon}{U_0}\right)^{\frac{a-d}{a}}. \quad (5.23)$$

The interaction of resonant pairs can be expressed similarly to Eqs. (5.19), (5.21) as

$$V_*^a = U_0 (n_{res}^a)^{\frac{a}{d}} = U_0 (P_0 T P_0 U_0)^{\frac{a}{d}} \left(\frac{\varepsilon}{U_0}\right)^{\frac{a-d}{d}}. \quad (5.24)$$

If  $(a-d)/d < 1$  the interaction dominates over the characteristic disordering energy  $\varepsilon$  at sufficiently small  $\varepsilon$ , which leads to the delocalization of excitations. In accordance with the previous analysis this always happens at  $a > 2d$ . Comparing the interaction Eq. (5.24) and the energy  $\varepsilon$  one can estimate the threshold energy. The estimate reproduces the previous result Eq. (5.19).

The energy  $E_*$  represents the system decoherence rate  $1/T_2 = \hbar/E_*$ . Indeed, this energy represents both the typical energy of interacting pairs and their typical interaction. This means that (1) the rate of pair irreversible dynamics induced by their interaction is given by the related time scale  $\tau_* \sim \hbar/E_*$ . This means that during the time  $\tau_*$  the transitions in resonant pairs create energy fluctuations of all TLS of order of  $E_*$ . Accordingly their phase fluctuation is given by

$$\delta\Phi(\tau_*) \sim \frac{\tau_* E_*}{\hbar} \sim 1 \quad (5.25)$$

which is the definition of a decoherence rate.

In the case of  $1/R^3$  interaction theory predicts that the decoherence rate is proportional to temperature which should dominate  $T^2$  dependence associated with the interaction of TLS with phonons at sufficiently low temperature. This expectation agrees with experimental observations of the linear temperature dependence of decoherence rate qualitatively. The parametric expression  $1/T_2 \sim T(P_0 U_0)^2/\hbar$  is smaller than the experimentally observed rate by the factor of 100 and this can be understood just incorporating missed factors coming from phase volume integration like  $(4\pi)$  which we skipped several times in our calculations. However the scaling theory that I described

cannot pretend on the numerical factor and therefore more accurate study is necessary. This study is very difficult because of the many character of interaction effects.

The general conclusion about inevitable delocalization of energy at arbitrary disorder for interaction decreasing with the distance as  $1/R^a$  with  $a < 2d$  should remain valid for low-dimensional system because the number of resonances rapidly increases after with the reduction of the relevant energy, which makes effects like coherent backscattering improbable. The effective phase space increases exponentially after the localization threshold is passed, which should eliminate all dimensional effects. Note that the results take place only at finite temperature while in the zero temperature limit, where the only single excitation problem is relevant there is no way to make flip-flop pairs responsible for energy delocalization. The threshold power of interaction there is  $a=d$  rather than  $2d$ . Analysis of more complicated clusters than pairs and triples does not change our estimates. When the energy is larger than the threshold energy  $E_*$  that contribution is small as  $E_*/E$ . Of course at  $E < E_*$  all contribution become important and nothing can be neglected. The system then should be treated differently.

It is interesting that the model of interacting two-level centers describes interacting qubits in quantum hardware. In many systems which can be used there the long-range  $1/R^3$  interaction is unavoidable. For example any electronic, Josephson or spin qubits inevitably have the long range dipole-dipole interaction  $1/R^3$ . This interaction can lead to decoherence that will break down quantum computations. This can be avoided however if the system is kept in external field working as effective disordering. The decoherence Eq. (5.21) takes place only if the number of interacting pseudospins is sufficiently large, meaning that the system size must exceed  $R_*(U_0 T_2 / \hbar)^{1/3}$ . Accordingly if the total number of qubits is sufficiently small  $N < n R_*^3$  the delocalization and decoherence will be substantially suppressed. Here  $n$  is the characteristic qubit density. Also the reduction of the effective system dimension remarkably reduces the destructive effect of the long-range interaction.

Our study addressed the decoherence rate. Under conditions of strong disordering the energy delocalization takes place only for the small fraction of TLS ( $P_0 U_0$ ), which are resonant pairs. Relaxation for TLS which are not in resonant pairs must also take place because interacting resonant pairs form the thermal bath capable to assist any transition having the finite transition amplitude. Energy delocalization makes such transitions irreversible. Below we study the relaxation rate in a 3-d system with  $1/R^3$  interaction. Analysis of other systems is beyond this class.

There are two possible scenarios of relaxation of arbitrarily TLS characterized by the energy  $E \sim T$  and tunneling amplitude  $\Delta_0 < E$ . It has neighboring thermal TLS, which can assist its transitions. One way is if it forms a pair with the other thermal TLS with energy  $E'$  and tunneling amplitude of order of  $E'$ , such that

$$|E - E'| < E_* \quad (5.26)$$

The characteristic distance  $R$  to such TLS can be defined using the phase volume estimate

$$P_0 E_* R^3 \sim 1 \rightarrow R \sim \left( \frac{1}{P_0 E_*} \right)^{\frac{1}{3}}. \quad (5.27)$$

The transition amplitude for such a pair is given by

$$\Delta_{0p} \sim \frac{U_0}{R^3} \frac{\Delta_0}{E} \sim E_* P_0 U_0 = T(P_0 U_0)^3, \quad (5.28)$$

and the transition rate can be estimated using the Landau Zener type expression

$$r_{p1} \sim \frac{\Delta_{0p}^2}{\hbar E_*} \sim \frac{T(P_0 U_0)^4}{\hbar}. \quad (5.29)$$

The inevitable relaxation of all thermal TLS results in spectral diffusion which varies all TLS energies within the energy domain ( $E - W_T, E + W_T$ ) where  $W_T \sim TP_0 U_0$  is the typical interaction of thermal TLS at neighboring distance. Then one can consider the alternative scenario of relaxation where the give TLS selects few neighboring TLS having energy  $E'$  such that

$$|E - E'| < W_T. \quad (5.30)$$

Then it can wait some characteristic time  $T_{sd}$  necessary for spectral diffusion to bring the pair energy into the resonant domain ( $-E_*, E_*$ ). Accordingly the transition rate (inverse relaxation time) for resonant TLS can be estimated as

$$r_1 = \frac{1}{T_1} \approx \frac{1}{T_{sd}}. \quad (5.31)$$

The relaxation of non-resonant TLS is more complicated. If the spectral diffusion passes the same energy domain in average once per the period (this seems to be the case for two-level systems) then the relaxation rate can be estimated as

$$r_1(E, \Delta_0) \approx \frac{1}{T_{sd}} \left( \frac{\Delta_0}{E} \right)^2. \quad (5.32)$$

Otherwise the dependence on tunneling amplitude can be weaker at intermediate tunneling amplitudes.

To study the spectral diffusion we can use the following model. Imaging that we study the relaxation of TLS pairs starting at the time  $t=0$  and investigate how many TLS pairs have transferred between their states during the time  $t$ . Imagine that the density of such pairs is  $x(t)$ . We are interested in thermal TLS pairs having tunneling amplitudes  $\Delta_0$  smaller than the resonant energy  $E_* \sim T(P_0 U_0)^2$ . Such a pairs are separated by the distance  $R$  such that

$$\Delta_0 \sim \frac{U_0}{R^3}. \quad (5.33)$$

The density of such TLS can be expressed as the product of thermal TLS density  $P_0 T$ , and the differential probability that the second TLS belongs to the spatial domain  $dR^3 = U_0 d\Delta_0 / \Delta_0^2$  and energy domain  $dE$ , so the density distribution of thermal TLS flip-flop pairs can be expressed as

$$P_0 T P_0 U_0 dE \frac{d\Delta_0}{\Delta_0^2}. \quad (5.34)$$

Thus the density distribution of resonant pairs over their energies and tunneling amplitudes can be written as (Burin, Kagan, 1995)

$$P_2(\Delta_0) = \frac{P_0}{\Delta_0} \frac{TP_0 U_0}{\Delta_0}. \quad (5.35)$$

Assume that  $W(\Delta_0, t)d\Delta_0$  is the density of TLS pairs which has transferred between their states in the time domain  $(0, t)$ . Then the total density of pairs which made transition in that time domain is given by

$$x(t) \approx \int_0^{E_*} d\Delta_0 W(\Delta_0, t). \quad (5.36)$$

Consider the time evolution of the density function  $W(\Delta_0, t)$ . Its time derivative is determined by pairs with the energy belonging to the resonant domain  $(-E_*, E_*)$  which did not transfer earlier in the time domain  $(0, t)$ . The density of such pairs can be expressed as

$$P_2(\Delta_0, t)E_* \quad (5.37)$$

where the function  $P_2(\Delta_0, 0)$  coincides with the “equilibrium” distribution Eq. (5.35). The transition probability for such pairs during the time  $dt$  can be expressed as

$$(\Delta_0 T_2)^2 \frac{dt}{T_2}. \quad (5.38)$$

We will express the function  $P_2(\Delta_0, t)$  through the function  $P_2(\Delta_0)$  multiplying it by the probability that this pair already made transition earlier at  $0 < \tau < t$ . The probability that the pair made transition during the infinitesimal interval  $d\tau$  can be expressed as (following by Fernandes and Alonso, see also Burin and Kagan, 1995)

$$W(0,0;t,\tau)E_*(\Delta_0 T_2)^2 \frac{d\tau}{T_2} \frac{dW(\Delta_0, \tau)/d\tau}{dW(\Delta_0, 0)/d\tau} \quad (5.39)$$

Here  $W(E_1, E_2; t, \tau)$  is the probability distribution for pair transition energies  $E_2$  at time  $\tau$  if it is equal to  $E_1$  at time  $t$ . This probability distribution can be expressed through the density of transferred TLS  $x$  as

$$W(E_1, E_2, t) = \frac{1}{\pi\eta U_0 x(t)} \frac{1}{1 + \left( \frac{E_1 - E_2}{\eta U_0 x(t)} \right)^2}, \quad \eta \sim 10. \quad (5.40)$$

The factor  $\frac{dW(\Delta_0, \tau)/d\tau}{dW(\Delta_0, 0)/d\tau}$  reflects the fact that only TLS, which make their first transition in the time domain  $(\tau, \tau+T_2)$  should be count. Then the total number of transitions will be given by the denominator, while the number of first time transitions is given by the numerator.

Finally the equation for the density of transferred spins with the given tunneling amplitude can be expressed as

$$\frac{\partial W(\Delta_0, t)}{\partial t} = \frac{(\Delta_0 T_2)^2}{T_2} \frac{P_0}{\Delta_0} \frac{TP_0 U_0}{\Delta_0} E_* \left[ 1 - \int_0^t \frac{d\tau E_*}{\pi\eta U_0 x(t) T_2} \frac{dW(\Delta_0, \tau)}{d\tau} \frac{\Delta_0^2 T_2}{P_0 TP_0 U_0 E_*} \right] \quad (5.41)$$

This equation can be rewritten in the form



$$\xi(\Delta_0, t) = A \left[ 1 - B \Delta_0^2 \int_0^t \frac{d\tau}{x(t)} \xi(\Delta_0, \tau) \right],$$

$$\frac{(\Delta_0 T_2)^2}{T_2} \frac{P_0}{\Delta_0} \frac{TP_0 U_0}{\Delta_0} E_* \left[ 1 - \int_0^t \frac{d\tau E_*}{\pi \eta U_0 x(t) T_2} (\Delta_0 T_2)^2 \xi(\Delta_0, \tau) \frac{\Delta_0^2 T_2}{P_0 TP_0 U_0 E_*} \right],$$

$$\xi(\Delta_0, t) = \frac{\partial W(\Delta_0, t)}{\partial t}, \quad (5.42)$$

$$A = P_0 U_0 P_0 T,$$

$$B = \frac{1}{\pi \eta U_0 E_*^2}.$$

Eq. (5.42) can be studied using the Laplace transform of both sides with respect to the time

$$F(\Delta_0, p) = \frac{A}{p} - \frac{AB\Delta_0^2}{p} \left( \frac{1}{x} \right)_p F(\Delta_0, p). \quad (5.43)$$

$$F(\Delta_0, p) = \int_0^{+\infty} dt e^{-pt} \xi(\Delta_0, t),$$

The solution to Eq. (5.43) is straightforward

$$F(\Delta_0, p) = \frac{A}{p \left[ 1 + AB\Delta_0^2 \left( \frac{1}{x} \right)_p \right]}. \quad (5.44)$$

Using the definition Eq. (5.36) one can obtain the self-consistent equation for the Laplace transform of the density of pairs  $x$

$$x_p = \int_0^{E_*} d\Delta_0 F(\Delta_0, p) = \frac{1}{p} \int_0^{E_*} \frac{Ad\Delta_0}{\left[ 1 + AB\Delta_0^2 \left( \frac{1}{x} \right)_p \right]} = \frac{\pi}{p} \sqrt{\frac{A}{B}} \frac{1}{\sqrt{\left( \frac{1}{x} \right)_p}}. \quad (5.45)$$

The upper integration limit in Eq. (5.45) can be approximately set to infinity in agreement with the result below. The self-consistent solution for time dependence of the density of transferred pairs can be expressed as

$$x(t) = \frac{\pi^2 A}{B} \frac{t}{\ln(t/T_2)} \approx 100 \frac{t}{T_2 \ln\left(\frac{t}{T_2}\right)} P_0 U_0 P_0 T. \quad (5.46)$$

The relaxation will take place when the density of transferred pairs approaches the density of thermal TLS which takes place at time

$$T_1 = \frac{\hbar}{T(P_0 U_0)^3}. \quad (5.47)$$

This estimate coincides with the estimate of the previous lecture obtained using the dynamical approach. Consequently for TLS with energy  $E$ , and tunneling amplitude  $\Delta_0$  one can express the relaxation rate as

$$r_1 = \frac{\Delta_0^2}{E^2} \frac{1}{T_1}. \quad (5.48)$$

Since the interaction  $1/R^3$  represents clearly the threshold regime for the self-supporting spectral diffusion the situation can be different for the interaction decreasing with the distance faster.

1. Burin, Natelson, Osheroff, Kagan, 1998
2. Black and Halperin