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SOLITONS WITH FERMION NUMBER $\frac{1}{2}$ IN CONDENSED MATTER AND RELATIVISTIC FIELD THEORIES

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States with fractional fermion charge have been discovered in relativistic field theory and condensed matter physics. In the latter context they lead to unexpected but experimentally verified predictions for one-dimensional electron-phonon systems like polyacetylene. We examine the common basis for this fortunate convergence between condensed matter and relativistic field theories.

1. Introduction

In a study of the spectrum for a one-dimensional, spinless Fermi field coupled to a broken symmetry Bose field, Jackiw and Rebbi (JR) [1] noted the occurrence of a localized zero-energy, c-number solution ψ_0 to the Dirac equation when a soliton is present. Furthermore, they proposed the interpretation that in the soliton-fermion system there is a twofold energy degeneracy (beyond the broken symmetry degeneracy), and that the two states carry charge $\pm\frac{1}{2}$. In other words, they found that introducing the soliton changes the number of fermions present by a fractional amount, namely $\pm\frac{1}{2}$, depending whether ψ_0 is occupied or not.

Independently, Su, Schrieffer and Heeger (SSH) [2], studying a coupled electron-phonon model for the quasi one-dimensional conductor polyacetylene $(CH)_x$, found a dynamical symmetry breaking of the system, which leads to degenerate vacua and soliton formation. In the presence of a soliton, there is a c-number solution ψ_0 of the electron field, localized near the soliton, with energy at the center of the gap. As a consequence, one-half a state of each spin orientation is removed from the sea in the vicinity of the soliton*. Thus, if one neglects the electron spin, the existence of the zero-energy state and fermion number $\pm\frac{1}{2}$ are common to the two situations.

In this paper, we outline the two theories and point out the fundamental reasons for similar behavior in different models. We hope that this will stimulate interaction between condensed matter and particle physicists.

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2. Solitons in a one-dimensional electron-phonon system

2.1. PRELIMINARIES

There are materials, such as polyacetylene $(CH)_x$, in which electrons move primarily in one dimension. Typically, these materials consist of parallel chains of atoms, or groups of atoms, in which electrons hop preferentially along the chains, while hopping between the chains is strongly suppressed. This anisotropy renders such materials quasi one-dimensional. The structure, called *trans*-$(CH)_x$, is shown in fig. 1.

In solids, the matrix element $t_{n'n}$, giving the probability amplitude of an electron hopping from site n to n', depends on the distance between these sites, whose average spacing is a. The atoms can be displaced from their perfect crystal lattice position for a variety of reasons, e.g. zero-point motions, thermal excitations, broken symmetry effects, etc. These displacements alter the matrix elements $t_{n'n}$, leading to the so-called electron-phonon (or electron-lattice displacement) interaction. Using the mean-field approximation, in which the phonon field u_n is treated as an unquantized c-number, Peierls [4] has shown that the one-dimensional electron-phonon system is unstable with respect to spontaneous breaking of the reflection symmetry $u_n \leftrightarrow -u_n$, for any non-zero electron-phonon coupling strength. The distortion gives rise to a charge density wave (CDW), in which the electron density and nuclear displacements oscillate periodically in space, with wave vector $\kappa = 2k_F$, where $\hbar k_F$ is the Fermi momentum of the valence electrons in the undistorted system. Bragg scattering of the electrons from the CDW potential opens a gap 2Δ, at the Fermi surface $\pm k_F$, in the electronic energy spectrum E; see fig. 2. For $(CH)_x$, there is exactly one electron per site, so $k_F = \pi/2a$. In the ground state, all one-electron states with $|k| < k_F$ are doubly occupied (spin up and down) leading to an insulator. In this case $\kappa = 2k_F = \pi/a$, and the CDW, with wavelength $2\pi/\kappa = 2a$, is commensurate with the lattice period. From the invariance of the system under discrete translations by $\pm a, \pm 2a, \ldots$, it follows that the ground state is twofold degenerate. This is so since a translation by $\pm a$ is equivalent to a reflection, but reflection symmetry is spontaneously broken. Consequently, states translated by a

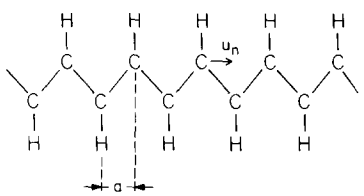

Fig. 1. *Trans* configuration of $(CH)_x$; $a = 1.2$ A. The coordinate displacement of the nth group, the phonon field, is denoted by u_n.

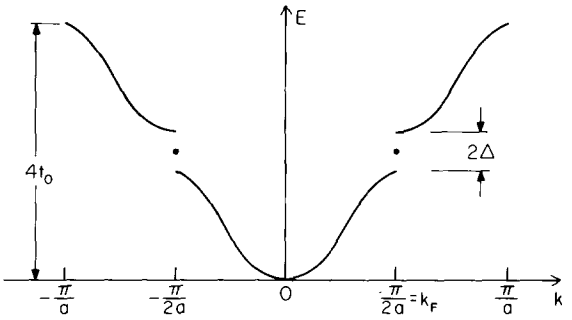

Fig. 2. Band structure due to the Peierls instability in $(CH)_x$.

are distinct from untranslated states. Furthermore, a soliton is formed by the domain boundary between the two ground states.

A one-dimensional, N-site, lattice hamiltonian for the above physical situation has the form

$$H=\sum_{n=1}^{N}\left(\frac{p_n^2}{2M}+V(u_n,u_{n+1})\right)-\sum_{\substack{n=1\\ s=\pm\frac{1}{2}}}^{N} t_{n+1\ n}\left(c_{n+1,s}^{\dagger}c_{n,s}+c_{n,s}^{\dagger}c_{n+1,s}\right). \quad (2.1)$$

Here u_n is a real, scalar, Bose field describing the coordinate displacement along the symmetry axis of the nth group (see fig. 1); p_n is the conjugate momentum, M being the group's mass (CH mass for polyacetylene). The first sum gives the phonon energy, kinetic and potential; the second, describes electron hopping from site n to site $n+1$, with amplitude $t_{n+1\ n}$. The fermion operators $c_{n,s}^{\dagger}$ and $c_{n,s}$ create and destroy electrons of spin $s(=\pm\frac{1}{2})$ at site n. In the SSH analysis, the potential energy is taken in a quadratic approximation,

$$V(u_n,u_{n+1})=\tfrac{1}{2}K(u_{n+1}-u_n)^2, \quad (2.2)$$

and the hopping amplitude is expanded to first order:

$$t_{n+1\ n}=t_0-\alpha(u_{n+1}-u_n). \quad (2.3)$$

For $(CH)_x$,

$$\alpha=4.1\ \text{eV}/\text{Å}, \qquad K=21\ \text{eV}/\text{Å}^2, \qquad t_0=2.5\ \text{eV}. \quad (2.4)$$

Thus the SSH hamiltonian is

$$H = \sum_{n=1}^{N} \left(\frac{p_n^2}{2M} + \frac{K}{2}(u_{n+1} - u_n)^2 \right)$$

$$- t_0 \sum_{\substack{n=1 \\ s=\pm\frac{1}{2}}}^{N} \left(c^\dagger_{n+1,s} c_{n,s} + c^\dagger_{n,s} c_{n+1,s} \right)$$

$$+ \alpha \sum_{\substack{n=1 \\ s=\pm\frac{1}{2}}}^{N} (u_{n+1} - u_n)\left(c^\dagger_{n+1,s} c_{n,s} + c^\dagger_{n,s} c_{n+1,s} \right). \tag{2.5}$$

The first sum in (2.5) is the harmonic vibrational energy of the free phonons; the second describes band electrons moving on a lattice in a tight binding approximation; the last sum provides a linear coupling of the phonon field to the electrons.

In the absence of interactions, $\alpha = 0$, the ground state is non-degenerate. It consists of the phonon vacuum and a Fermi electron sea, with all states doubly occupied up to $k = |k_F|$. In this limit the gap parameter vanishes, $\Delta = 0$.

2.2. BROKEN SYMMETRY GROUND STATE

To understand the symmetry breaking, it is convenient to introduce a staggered displacement field

$$\phi_n = (-1)^n u_n. \tag{2.6}$$

The ground state is determined by making an adiabatic (Born-Oppenheimer) approximation: the phonon's kinetic energy is ignored, and ϕ_n is set to a constant value u. The ground-state energy is determined as a function of u, and one finds $E_0(u) = E_0(-u)$; finally, $E_0(u)$ is minimized with respect to u. A non-vanishing minimization value for u signals spontaneous breaking of the reflection symmetry; thus, if u_0 is one such value, so is $-u_0$. The adiabatic approximation is justified *a posteriori* by noting that in the broken symmetry ground state, the typical phonon energy $\sqrt{K/M}$ is much smaller than the gap 2Δ in the electron spectrum, for interesting values of α. A straightforward calculation gives [2]

$$\frac{E_0(u)}{N} = -\frac{4t_0}{\pi} I\left(1 - \frac{4\alpha^2 u^2}{t_0^2}\right) + 2Ku^2, \tag{2.7}$$

where I is the elliptic integral. $E_0(u)$ is plotting in fig. 3. Two minima are seen at $u = \pm u_0$, corresponding to doubly degenerate ground states, A and B; while $u = 0$ is a local maximum, thus verifying Peierls' theorem. With numerical values for the

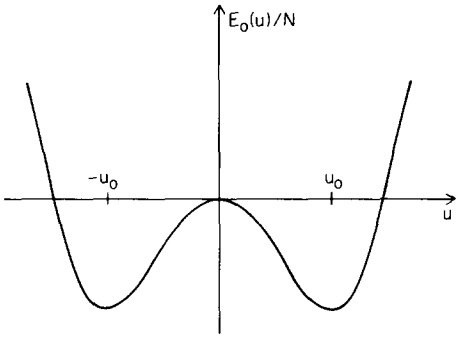

Fig. 3. Born-Oppenheimer energy per CH group, plotted as a function of the staggered displacement field $u = \phi_n = (-1)^n u_n$. The two stable minima correspond to A$(+u_0)$ and B$(-u_0)$ phases.

parameters appropriate to $(CH)_x$, one finds $u_0 = 0.042$ Å. Because of the periodic spatial variation of the electron hopping matrix element when $u \neq 0$, the electronic plane wave k of the undistorted chain mixes with the state $k - \kappa$ (or $k + \kappa$ if $k < 0$), leading to a gap 2Δ in the electron spectrum, as shown in fig. 2, where $\kappa = \pi/a$ is the first reciprocal lattice vector of the broken symmetry lattice. One finds

$$\Delta = 4\alpha|u_0|. \tag{2.8}$$

2.3. SOLITON EXCITATIONS

As a consequence of the ground state's twofold degeneracy ($\phi_n = \pm u_0$), there exist topological solitons, which act as boundaries between domains having different ground states. It is convenient to work with a chain formed as a large ring, having an even number of atoms; see fig. 4. Let the staggered field be plotted radially, with $\phi_n > 0$ outside the ring and $\phi_n < 0$ inside. For ϕ_n to be single valued, it must change sign an even number of times, in going around the ring. Thus when ϕ_n interpolates between the two vacua, we are led to consider a soliton S and an antisoliton $\bar{\text{S}}$, located at sites n_1 and n_2, respectively, and we take them to be widely separated. To find the shape of ϕ_n which describes the solitons, we need to minimize the adiabatic energy, subject to the constraint that ϕ_n approaches the A vacuum $(+u_0)$ and the B vacuum $(-u_0)$ far from S and $\bar{\text{S}}$, as illustrated in fig. 4. A numerical calculation yields [2]

$$\phi_n = u_0 \tanh\big[(n - n_1)/l\big] - u_0 \tanh\big[(n - n_2)/l\big] - u_0, \tag{2.9}$$

where the soliton width l is approximately 7 for $(CH)_x$. Near the solitons, the above reduces to the familiar topological kink, with hyperbolic tangent profile, which is appropriate for a double-well potential of the ϕ^4 field theory [see (3.3)]. The energy of this solution, interpreted as the soliton creation energy, is found to be $E_S = 0.42$ eV $\simeq 0.6\Delta$.

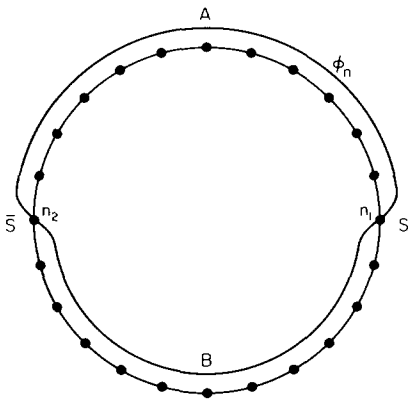

Fig. 4. Soliton S and antisoliton $\bar{S}$ occurring in a ring with an even number of atoms. The staggered field ϕ_n is plotted radially.

In a continuum version of the SSH model, Takayama, Lin-Liu and Maki (TLM) [5] have shown that the hyperbolic tangent exactly satisfies the coupled mean field equations ($M \to \infty$). They find $E_S = 2\Delta/\pi \simeq 0.63\Delta$, in good agreement with the results of the discrete calculation. If the coupling constant α is increased, the soliton width l decreases and discrete lattice effects become increasingly important.

The electronic spectrum in the presence of S and $\bar{S}$ can be readily determined, both in the lattice and in the continuum models. One finds that the change of the electronic state density $\rho(E)$ in the presence of solitons relative to that in their absence exhibits two discrete states $\psi_{0\pm}$, whose energies are symmetrically located about the center of the gap, $E = 0$. As $|n_1 - n_2| \to \infty$, the energy splitting between the two states vanishes as $e^{-|n_1 - n_2|/l}$, so that one can form zero-energy eigenstates ψ_{0S} and $\psi_{0\bar{S}}$, localized about S and $\bar{S}$, as linear combinations of the two states $\psi_{0\pm}$. ψ_{0S} and $\psi_{0\bar{S}}$ are given by [2]

$$\psi_0(n) = \frac{1}{l^{1/2}} \operatorname{sech}\left[(n - n_i)/l\right] \cos\left[\tfrac{1}{2}\pi(n - n_i)\right],$$
$$S: i = 1, \qquad \bar{S}: i = 2. \tag{2.10}$$

To see how the continuum states are altered by S and $\bar{S}$, we note that the completeness of electronic eigenstates ψ_ν, at each site, implies that the energy integral of the local density $\rho_{n\,n}(E)$, at any site n, is unity:

$$\int_{-\infty}^{\infty} dE\, \rho_{n\,n}(E) = 1,$$
$$\rho_{n\,n}(E) = \sum_\nu |\psi_\nu(n)|^2 \delta(E - E_\nu),$$
$$\rho(E) = \sum_{n=1}^{N} \rho_{n\,n}(E). \tag{2.11}$$

Also the electronic hamiltonian is odd under charge conjugation, $c_n^\dagger \rightleftharpoons c_n$, so that

$$\rho_{n\,n}(E) = \rho_{n\,n}(-E). \tag{2.12}$$

Therefore, breaking up the spectral integral in eq. (2.11) into contributions from the negative and positive energy continua plus the discrete zero-energy state, and using the symmetry (2.12), gives

$$2\int_{-\infty}^{0} \mathrm{d}E\rho'_{n\,n}(E) + |\psi_0(n)|^2 = 2\int_{-\infty}^{0} \mathrm{d}E\rho_{n\,n}(E). \tag{2.13}$$

Here the primed quantity is the local density in the presence of the soliton; the unprimed, in the vacuum where the soliton is absent. Thus, the local deficit in negative energy states satisfies

$$\int_{-\infty}^{0} \mathrm{d}E\left[\rho'_{n\,n}(E) - \rho_{n\,n}(E)\right] = -\tfrac{1}{2}|\psi_0(n)|^2. \tag{2.14}$$

Summing over all sites, and using the normalization condition on ψ_0, one finds that the total deficit from the negative energy valence sea is precisely $\frac{1}{2}$ a state per spin. Including spin, a total of one electron is missing from the Fermi sea, due to the soliton. If ψ_0 is unoccupied, all spins are paired and the soliton has charge $Q = +e$ and spin $s = 0$. Correspondingly, when ψ_0 is singly occupied, the soliton is neutral $Q = 0$, but the spin is $\pm\frac{1}{2}$. The two states are degenerate in energy, since ψ_0 is a zero-energy eigenstate, for infinite separation between S and $\bar{\mathrm{S}}$. (However, one does not usually compare these two situations since they involve a different number of electrons.)

These local charge-spin relations would appear to violate Kramer's theorem, since one cannot go from integer spin to integer spin and remove or add one electron with spin $\frac{1}{2}$. Nevertheless, these peculiar charge-spin relations have been observed experimentally in $(CH)_x$ *. Conventional fermion excitations in solids (electrons and holes) have charge $\pm e$ and spin $\pm\frac{1}{2}$. The resolution of the apparent paradox is that the antisoliton also has these spin-charge relations, so the difficulty is removed by the topological requirement that S and $\bar{\mathrm{S}}$ be created in pairs, even though they act as independent excitations when widely separated. In other words, global constraints relating charge and spin are valid, but they do not fix local charge-spin relations.

3. Fermionic solitons in a relativistic field theory

3.1. PRELIMINARIES

In a study of the spectrum for a Dirac field coupled to a broken symmetry Bose field, JR discovered in various models the existence of zero-energy fermion eigen-

* For evidence of charged, spinless solitons, see ref. [6a]. Evidence for neutral, spin-$\frac{1}{2}$ solitons is contained in ref. [6b].

states, localized in the vicinity of a soliton of the Bose field. Their model, involving a scalar field Φ coupled to a Fermi field Ψ in one continuous spatial dimension, is closely related to the above discussed solutions from condensed matter theory.

In the one-dimensional model, JR consider a hamiltonian [1]

$$H = \int \mathrm{d}x \left\{ \tfrac{1}{2}\Pi^2 + \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\Phi \right)^2 + V(\Phi) + \Psi^\dagger(\alpha p + \beta g\Phi)\Psi \right\},$$

$$p = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x}. \tag{3.1}$$

Here Π is the momentum conjugate to Φ, and $V(\Phi)$ is a potential energy density for Φ. The Dirac spinors have two components, hence the fermions carry no spin—a simplifying option available in one dimension. Correspondingly, the two Dirac matrices α and β are two-dimensional:

$$\alpha = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \beta = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.2}$$

The model is analyzed in an adiabatic approximation*. The ground state is determined by minimizing $\int \mathrm{d}x[\frac{1}{2}((\mathrm{d}/\mathrm{d}x)\Phi)^2 + V(\Phi)]$ with constant Φ, while the soliton is found by minimizing with an x-dependent Φ, subject to the condition that the total energy be finite. The Dirac hamiltonian is then quantized, with Φ taken as an external, prescribed, c-number, background field. For present purposes $V(\Phi)$ is a symmetric, double-well potential, as in fig. 3, but of simpler analytic form:

$$V(\Phi) = \frac{\lambda^2}{2\mu^2}(\mu^2 - \Phi^2)^2 = V(-\Phi). \tag{3.3}$$

3.2. BROKEN SYMMETRY GROUND STATE

The minimum of (3.3) is at $\Phi = \pm\mu$; thus the reflection symmetry $\Phi \leftrightarrow -\Phi$ is spontaneously broken. The Dirac hamiltonian describes a free particle with mass gap $\Delta = g\mu = m$. The fermion modes satisfy the free Dirac equation

$$(\alpha p + \beta m)u^{(\pm)} = \pm|\varepsilon|u^{(\pm)},$$

$$|\varepsilon| = \sqrt{k^2 + m^2}. \tag{3.4}$$

Charge conjugation is implemented by the σ^3 matrix. It takes positive energy solutions of (3.4) into negative energy solutions, and vice versa. Quantization is

* It can be shown that this approximation is the starting point for a systematic, weak coupling expansion; see ref. [7].

achieved by the usual expansion in modes:

$$\Psi = \sum_k \left\{ e^{-i\varepsilon_k t} b_k u_k^{(+)}(x) + e^{i\varepsilon_k t} d_k^\dagger v_k^{(-)}(x) \right\}. \tag{3.5}$$

Here $u_k^{(+)}$ is a positive energy solution, and $v_k^{(-)}$ is the charge conjugate of the negative energy solution:

$$v_k^{(-)} = \sigma^3 \left(u_k^{(-)} \right)^*,$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.6}$$

The operators $b_k^\dagger$ (b_k) create (annihilate) particles, while $d_k^\dagger$ (d_k) do the same for antiparticles. The charge operator

$$Q = \int dx : \Psi^\dagger(x)\Psi(x) :$$

$$= \tfrac{1}{2} \int dx \sum_{i=1}^{2} \left(\Psi_i^\dagger(x)\Psi_i(x) - \Psi_i(x)\Psi_i^\dagger(x) \right), \tag{3.7}$$

becomes

$$Q = \sum_k \left(b_k^\dagger b_k - d_k^\dagger d_k \right). \tag{3.8}$$

The spectrum is elementary and can be built on either the A vacuum ($\Phi = \mu$) or the B vacuum ($\Phi = -\mu$). Of course the vacua are charge neutral,

$$Q|0\rangle = 0. \tag{3.9}$$

3.3. SOLITON EXCITATIONS

The existence of two ground states leads to topological solitons, which interpolate between them. To find the soliton shape of Φ, we solve the equation

$$-\frac{d^2}{dx^2}\Phi(x) + V'(\Phi) = 0. \tag{3.10}$$

For (3.3), finite energy solutions are given by*

$$\Phi(x) = \pm\mu \tanh \lambda x. \tag{3.11}$$

* The origin has been arbitrarily set at $x = 0$, the location of the soliton. Of course, the soliton may be located at any $x = x_0$, and the proper quantum mechanical treatment of this degree of freedom is given in ref. [7]. For a review, see ref. [8].

Note that as x passes from negative infinity to positive infinity, $\Phi(x)$ interpolates between the two vacua. With the positive sign, (3.11) describes the soliton S; with the negative, the antisoliton $\bar{\mathrm{S}}$. The energy of the solution (3.11), interpreted as the soliton formation energy, is

$$E_{\mathrm{S}} = \int \mathrm{d}x \left\{ \frac{1}{2} \left(\frac{\mathrm{d}}{\mathrm{d}x} \Phi \right)^2 + V(\Phi) \right\}$$

$$= \int \mathrm{d}x \left(\frac{\mathrm{d}}{\mathrm{d}x} \Phi \right)^2 = \tfrac{4}{3} \mu^2 \lambda . \tag{3.12}$$

The quantization of the Dirac equations, proceeds as in the vacuum sector, except that modes in the presence of the soliton satisfy

$$(\alpha p + \beta m \tanh \lambda x) U^{(\pm)} = \pm |\mathcal{E}| U^{(\pm)} . \tag{3.13}$$

Again charge conjugation insures that the positive eigenvalue is paired with a negative one, and the modes may be explicitly found. Additionally, (3.13) admits a zero-eigenvalue solution,

$$\psi_0(x) \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\left(-m \int_0^x \mathrm{d}x' \tanh \lambda x' \right), \tag{3.14}$$

which is charge conjugation self-conjugate,

$$\psi_0^{\mathrm{c}}(x) = \sigma_3 \psi_0^*(x) = \psi_0(x) . \tag{3.15}$$

Thus the spectrum exhibits at the center of the gap an additional state.

Quantization again proceeds by an expansion in modes,

$$\Psi = a \psi_0(x) + \sum_k \left[\mathrm{e}^{-i \mathcal{E}_k t} B_k U_k^{(+)}(x) + \mathrm{e}^{i \mathcal{E}_k t} D_k^\dagger V_k^{(-)}(x) \right], \tag{3.16}$$

$U_k^{(+)}$ being a positive energy solution, and $V_k^{(-)}$ the charge conjugate of the negative energy solution,

$$V_k^{(-)} = \sigma^3 \left(U_k^{(-)} \right)^* . \tag{3.17}$$

The operators $B_k^\dagger$ (B_k) and $D_k^\dagger$ (D_k) create (annihilate) conventional fermions and antifermions in the soliton sector. However, the further operator a when operating on the soliton state produces another state of the same energy; hence the two states

are degenerate in energy. To distinguish them, we may label them as $|\pm, S\rangle$ and

$$a|+,S\rangle = |-,S\rangle,$$

$$a^\dagger|-,S\rangle = |+,S\rangle,$$

$$a|-,S\rangle = 0,$$

$$a^\dagger|+,S\rangle = 0. \tag{3.18}$$

In analogy with the physical situation encountered in the polyacetylene system, we may call the "plus" state occupied and the "minus" state unoccupied.

The charge quantum number of $|\pm, S\rangle$ is evaluated by substituting the expansion (3.16) into (3.7). One finds

$$Q = a^\dagger a - \tfrac{1}{2} + \sum_k \left(B_k^\dagger B_k - D_k^\dagger D_k \right). \tag{3.19}$$

Consequently it follows that

$$Q|\pm, S\rangle = \pm\tfrac{1}{2}|\pm, S\rangle, \tag{3.20}$$

i.e., each of the two soliton states carry $\frac{1}{2}$ unit of charge.

4. Discussion

The two models under discussion—one drawn from a realistic situation in condensed matter physics, the other from a formal, mathematical investigation in relativistic quantum field theory—obviously differ in detail. The SSH hamiltonian is on a lattice, and also the fermions carry a spin degree of freedom. The JR hamiltonian is in the continuum; the fermions are spinless. A closer comparison can be made with the TLM hamiltonian—a continuum approximation to SSH, if the fermion spin is ignored for simplicity. The structures of the JR and TLM fermion hamiltonians coincide, but differences remain in the boson parts.

Nevertheless, in crucial respects the two models are similar: both give rise to spontaneous breaking of the field reflection symmetry, and as a consequence, have doubly degenerate ground states. They possess soliton excitations which interpolate between the degenerate vacua. Moreover, the fermion equation in both cases admits a localized zero-energy solution, which then implies charge fractionalization: $\frac{1}{2}$ unit of charge is gained or lost depending whether ψ_0 is filled or empty. In the condensed matter example, the fractionalization is obscured by a doubling of degrees of freedom due to spin, but an experimentally observed signal remains in unusual charge-spin relations: charged solitons are spinless and neutral solitons carry spin $\frac{1}{2}$.

In both models the presence of charge conjugation symmetry allows the pairing of positive energy modes with negative energy modes, leaving the zero-energy state unpaired.

The common phenomena are in fact universal. It has been shown under very general mathematical hypotheses, that the Dirac equation in the background field of a topologically interesting configuration, like a soliton, always possesses zero-eigenvalue modes whose number is related to an integer which characterizes the non-trivial topology*. Since the result is valid in any number of dimensions, the possibility exists that effects similar to those observed in polyacetylene may be found in higher-dimensional systems, e.g. associated with vortices in superfluid helium three.

The SSH-TLM model is also interesting for particle physicists in that it realizes its symmetry breaking due to the Peierls instability through quantum-mechanical dynamics. In the classical, tree approximation where fermions are ignored, the only dynamics for the bosons are the harmonic lattice vibrations; in contrast to the JR hamiltonian, for which even in the absence of the fermions and without quantal effects for the bosons, the classical solutions break the symmetry. The effective potential of fig. 3 for the Bose field is generated dynamically in the condensed-matter application, while it is arbitrarily posited in the mathematical field theory example. The possibility of realizing spontaneous symmetry breaking by dynamics, rather than by assumption, is widely discussed but rarely achieved by particle physicists.

It is truly remarkable that a phenomenon as esoteric and peculiar as charge fractionalization should have been discovered in two different contexts: mathematical investigations of model field theories by particle physicists; description of experimental phenomena by condensed matter physicists. That this should happen is strong reaffirmation of the unity of all branches of physics and another example of the power of mathematics to uncover unexpected physical behavior.

The ideas on charge fractionalization can be carried further. It has been suggested that arbitrary charge fractions can be obtained in fermion-soliton systems, provided charge conjugation is abandoned and the vacuum structure is sufficiently complex. An example with $\frac{1}{3}$ units of charge, which is not obscured by the two spin states, has been discussed by Su and Schrieffer [10] in the condensed matter context of TTF-TCNQ; while related ideas for particle physics have been investigated by Goldstone and Wilczek [11].

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* A review of these "index theorems" is found in ref. [9].

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