EUROPHYSICS LETTERS 15 March 1991

Europhys. Lett., 14 (6), pp. 569-574 (1991)

Quasi-Particle Quantum Numbers in Two and Three Dimensions.

P. C. E. STAMP

Physics Department, University of British Columbia
6224 Agricultural Rd., Vancouver, B.C., Canada V6T 2A6

(received 18 July 1990; accepted in final form 11 January 1991)

PACS. 67.40D – Quantum statistical theory; ground state, elementary excitations.
PACS. 67.50D – Normal phase.
PACS. 73.20D – Electron states in low-dimensional structures (inc. quantum wells, superlattices, layer structures and intercalation compounds).
PACS. 74.70T – Heavy-fermion superconductors.

Abstract. – It is shown how quasi-particle quantum numbers may be defined and calculated for interacting Fermi systems in 2 and 3 dimensions. Exact results are given for charged and neutral Fermi systems, both normal and superfluid, in 3 dimensions, and for Cherns-Simons implementations of anion theories in 2 dimensions. The latter is applied to the fractional Hall effect. In all cases, the local quasi-particle quantum numbers vary continuously with interactions and/or temperature.

There has been great interest recently in the possibility of exotic quasi-particle states in 2-dimensional Fermi systems. Such states are already known to exist in 1-dimensional systems [1, 2], and in the fractional quantum Hall effect (FQHE) [3], and many recent theories of high-T_c superconductivity depend crucially on the existence of quasi-particles with precisely defined fractional quantum numbers [4].

However the definition and calculation of these quantum numbers (or «charges») turn out to be full of surprises, even for 3-dimensional systems. Here a way of calculating both «local» and «global» charges will be given, along with exact results for a variety of systems. Apart from suggesting a number of interesting experiments, these calculations also considerably clarify the issues at stake in the discussion of exotic quasi-particles.

Definition of quasi-particle charges. – Consider some 2- or 3-dimensional system composed of interacting fermions, and eigenstates labelled by quantum numbers $\{\xi_j\}$. The expectation value of some local operator $\hat{X}(r, t)$ acting on the system in a state $|\alpha\rangle$, with one single quasi-particle, is $\langle \hat{X}_\alpha(r, t)\rangle = \langle \alpha|\hat{X}(r, t)|\alpha\rangle$. The Fourier transform $\langle \hat{X}_\alpha(Q)\rangle$ of this $\Lambda_\alpha^X(Q)$, the fully renormalized 3-point vertex describing interactions between the normalized «quasi-particles» and the field $X(Q)$ (here $Q = (q, \omega)$). In general we shall deal with quasi-particle wave-packets $|X\rangle$, which can nevertheless be labelled using the conserved quantities $\{\xi_j\}$ of the system.

We now define the functions $\bar{X}_\alpha(t)$ for different «charges» as

$$\bar{X}_\alpha(t) = \int \mathrm{d}r^D \theta(R - |r|) \langle X_\alpha(r, t) \rangle , \tag{1}$$

where the system size $L \gg R$, and we require $R \gg t\Delta p/m$, the free-particle wave-packet spread after time t (with momentum spread Δp); we also require $\Delta r(t=0) \ll R$. The «local quasi-particle charges» are given by $X_\alpha^{\text{loc}} = \bar{X}_\alpha(t \to \infty)$ (but still keeping $R \gg t\Delta p/m$, in this long-time limit), while the «global quasi-particle charges» $X_\alpha^{\text{glob}} \equiv \bar{X}_\alpha(t \to 0)$. Thus we see that the global charge X_α^{glob} refers to the expectation value of X_α averaged over the entire system (or over a small part of it at short times). However the local charge X_α^{loc} refers to that part of this charge that «stays together», in a somewhat distorted and slowly spreading «packet», as time goes on. Note that the shape and size of this packet (which is really a density matrix) is different for each different quantum number (see below). The difference between X_α^{loc} and X_α^{glob} arises solely from interactions.

3-dimensional systems. – It is very useful to start by considering some familiar examples. A *neutral* 3-dimensional Fermi liquid has 1-quasi-particle states $|p\sigma\rangle$, for which

$$X_{p\sigma}(Q) = \lambda_{\sigma\sigma'}^X \sum_{p'\sigma'} \left[\delta_{pp'} + \left(\frac{\boldsymbol{q} \cdot \boldsymbol{v}_{p'\sigma'}}{\boldsymbol{q} \cdot \boldsymbol{v}_{p'\sigma'} - \omega} \right) T_{p'p}^{\sigma'\sigma}(Q) \right] \tag{2}$$

(we consider wave-packets below). Here $\lambda_{\sigma\sigma'}^X$ is the bare 3-point vertex for quasi-particle interactions with the field X and $T_{pp'}^{\sigma\sigma'}(Q)$ is the *renormalized on-shell* quasi-particle T-matrix [5]. We assume that our initial quasi-particle energy $\varepsilon_{p\sigma}$ is considerably less than the typical fluctuation energies of the system (note $\varepsilon_{p\sigma}$ is a complex function of $\zeta_{p\sigma} = (p - p_{\mathrm{F}}^\sigma) v_{\mathrm{F}}^\sigma$; and $\varepsilon_{p\sigma} = \zeta_{p\sigma}$ for very low $\varepsilon_{p\sigma}$ [5]).

We may then solve (2) using microscopic Fermi-liquid theory, in terms of the Landau parameters F_l^S, F_l^A. The techniques are standard [5, 6], but the results are actually rather surprising. Considering for example the fermion number density $n_p(Q)$, and taking only $l = 0$, 1 parameters (as for ^{3}He liquid), one finds that

$$\langle n_p(Q) \rangle = \frac{1 + F_1^S[1/3 + \chi_0(\eta)(\eta^2 - \eta(\cos\theta_p - \chi_0(\eta)\cos^2 2\theta_p))]}{[1 + F_0^S \chi_0(\eta)][1 + F_1^S(1/3 + \eta^2 \chi_0(\eta))] - \eta^2 F_0^S F_1^S \chi_0^2(\eta)} + O(\zeta_p^2 \ln \zeta) , \tag{3a}$$

$$\chi_0(\eta) = 1 - \frac{\eta}{2} \ln \left| \frac{1+\eta}{1-\eta} \right| + \imath \frac{\pi}{2} \eta \theta(1 - |\eta|) , \tag{3b}$$

where $\eta = \omega/qv_{\mathrm{F}}$, and $\theta_p = \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{q}}$. This very complex result contains all the details of the «decay down» of $|p\sigma\rangle$, via particle/hole and collective mode emission(¹). However although the Fourier transform is also very unwieldy (it is in fact the generalization of the 1st-order calculation of ref. [7] to all orders in perturbation theory), the long- and short-time results are very simple. Thus one finds $n_p^{\text{glob}} = 1$, whilst $n_p^{\text{loc}} = 1/(1 + F_0^S)$; and analogous calculations for spin and current give $\boldsymbol{S}_{p\sigma}^{\text{glob}} = (1/2)\gamma\hbar\boldsymbol{\sigma}$, $\boldsymbol{J}_p^{\text{glob}} = \boldsymbol{p}/m$, but $\boldsymbol{S}_{p\sigma}^{\text{loc}} = (1/2)\gamma\hbar\boldsymbol{\sigma}/(1 + F_0^A)$, and $\boldsymbol{J}_p^{\text{loc}} = \boldsymbol{p}/m(1 + 1/3F_1) = \boldsymbol{p}/m^*$. The difference between the global and local results describes «charge» that has escaped to (or been sucked in from) infinity. These fractions differ for each charge/quantum number, so that we have, *e.g.*, «partial spin/charge separation» at long times. Lest the reader doubt the applicability of our definitions here, it should be noted that

(¹) The details of the calculations of $\hat{X}_\alpha(Q)$ and its Fourier transform $\hat{X}_\alpha(r, t)$ are technically interesting but very lengthy, and will be given in a longer paper.

the functions $\langle n_p(Q)\rangle$, $\langle S_{p\sigma}(Q)\rangle$, etc., are nothing but the Landau distribution functions for density, spin, etc., since $\Lambda_{p\sigma}(Q)$ solves the Landau-Boltzmann equation [6]. Thus our definitions of local and global quasi-particle charges correspond simply to the local and global parts of the relevant Landau distribution functions—which are themselves simply expectation values $\mathrm{Tr}\{\rho\hat{X}_\alpha\}$ over the reduced density matrix ρ [5].

These results are easily generalized to globally neutral electronic systems, but with one subtlety. At very short times, the standard calculation of the 3-point vertex gives $\Lambda_{p\sigma}(Q)=\varepsilon^{-1}(Q)\widetilde{\Lambda}_{p\sigma}(Q)$, where $\widetilde{\Lambda}_{p\sigma}(Q)$ is the «proper 3-point vertex» not containing direct Coulomb lines [6]. Now $\Lambda_{p\sigma}(Q)$ describes a very localized electronic wave-packet, whose electric charge is *not* locally compensated. But the correct description of the quasi-particles at long times is given by $\widetilde{\Lambda}_{p\sigma}(Q)$, which satisfies the Landau-Silin equation; as is well known, this function describes, at long times, fermionic charge spread *uniformly*, thereby preserving *local* charge neutrality (cf. ref. [6]). Of course if we added electrons to the system, uncompensated by neutralizing charge, they would go to the walls [8]; but it is quite wrong to associate such excitations with quasi-particles, as usually defined.

Partial spin/charge separation also occurs—a fraction $F_0^A(1+F_0^A)^{-1}$ of the spin «escapes to infinity». F_0^A can be extracted from spin-wave measurements.

It is often assumed that the sharpness of quasi-particle charges may be restored if there are no gapless excitations. While this is often true in 1 dimension, it is incorrect in 3 dimensions. Consider, *e.g.*, a general singlet neutral superfluid. For short times one finds the usual results $n_p^{\mathrm{glob}}=(|u_p|^2-|v_p|^2)$, $\boldsymbol{J}_p^{\mathrm{glob}}=\boldsymbol{p}/m$, and $\boldsymbol{S}_{p\sigma}^{\mathrm{glob}}=(1/2)\,\gamma\hbar\boldsymbol{\sigma}$, and straightforward generalization of the method given above yields the long-time limits

$$\begin{cases} n_p^{\mathrm{loc}}=(\,|\,u_p\,|^2-|\,v_p\,|^2)\,Y(T)(1+F_0^S\,Y(T))^{-1}\,,\\[1ex] \boldsymbol{S}_{p\sigma}^{\mathrm{loc}}=\dfrac{1}{2}\,\gamma\hbar\boldsymbol{\sigma}Y(T)(1+F_0^A\,Y(T))^{-1}\,,\\[1ex] J_{p_i}^{\mathrm{loc}}=\left[\dfrac{Y(T)}{1+1/3F_1^S\,Y(T)}\right]_{ij}\dfrac{P_j}{m}\,,\end{cases} \tag{4}$$

where $\hat{Y}_{ij}(T)$ and $Y(T)$ are the matrix and scalar Yosida functions [9] for the appropriate gap function (*s*-wave, *d*-wave, etc.). Again partial (and only partial) spin/charge separation occurs. Moreover this partial separation is *not* changed, if we add Coulomb interactions to the system—exactly as for the metal described above, quasi-particles are neutral in the long-time limit, and $\boldsymbol{S}_{p\sigma}^{\mathrm{loc}}$ is still given by (4). Thus it is incorrect to regard the quasi-particles in 3-d superconductors as spinons [8].

In view of these results, one is led to ask how to properly define quasi-particle *statistics*. In 3-d systems this is normally done quite unambiguously via their *global* commutation relations [6]. This is equivalent to the global fermionic charge defined above, which is equal to unity for fermionic systems. The *local* fermionic charge n_p^{loc} is not the same. In fact it corresponds to the Berry phase ϕ_p that one would obtain by slowly moving one quasi-particle around a second one, on a circle of radius R centered on the second (and with $R\gg t\,\Delta p/m$). The demonstration that $\phi_p=2\pi n_p^{\mathrm{loc}}$ is then essentially the same as that in the anion literature (see, *e.g.*, Arovas *et al.* [10]), since the excess phase accumulated corresponds to the excess enclosed fermionic charge. However in 3 dimensions this Berry phase definition is somewhat artificial (since we can always deform the circle into a quite different curve, with a different ϕ_p), so it is best to stick to the definition of n_p^{loc} given previously.

2 *dimensions.* – Elementary consideration of the 2-particle scattering matrix for point particles shows that in 3 dimensions it is forced by undistinguishability and unitarity to take the form $K(\Omega) = f(\boldsymbol{\Omega}) \pm f(\boldsymbol{\Omega} + \pi)$. However in 2-d the more general

$$K(\theta) = \sum_{n=-\infty}^{\infty} \exp[2i\alpha n] f(\theta + 2\pi n)$$

is allowed [10], yielding «anions», with statistics and fermionic charge α. This result follows for point particles when $\alpha \neq 0$ because the diverging centrifugal force (as $|\boldsymbol{r}_1 - \boldsymbol{r}_2| \to 0$) prevents world lines from crossing. But how do we deal with *quasi-particles*, which are always smeared out in space?

A common answer to this is to argue that, if the quasi-particles are widely separated, then the above argument (or its more rigorous braid group formulation [10]) is still applicable, since world lines will then rarely cross. The argument would then justify *a posteriori* the use of Berry's phase to define quasi-particle statistics in, *e.g.*, the fractional Hall effect (FQHE); it gives anions with fractional fermionic charge $n^{\text{loc}} = \nu = \pm 1/(2l+1)$, where $l = 1, 2, \ldots$, and ν is the Landau level filling fraction. At temperature $T = 0$ this result follows from Laughlin's wave function [3], and is easily shown using the methods above, since the charge does not spread at $T = 0$. Hence we find [3, 10] that $\phi_p = 2\pi n_p^{\text{loc}} = 2\pi\nu$.

However at finite T things are more subtle. It has not yet been possible to generalize the Laughlin theory to finite T, but we can resort to the effective action theories that have been recently devised [11]. The simplest versions of these have a Lagrangian density

$$L(r,t) = \Psi^{+}\left[(i\partial_t - e(A_0 + a_0)) - \frac{K}{2}(\nabla - i(e\boldsymbol{A} + \boldsymbol{a}))^2\right]\Psi + \beta|\Psi|^2 - \lambda|\Psi|^4 - \frac{e^2\nu}{4\pi}\varepsilon^{\lambda\nu\sigma} a_\lambda \partial_\nu a_\sigma, \quad (5)$$

where the fields $\Psi(r,t)$ can be interpreted, following Read [11], as the amplitude for finding a particle at $(\boldsymbol{r}, t)$. At $T = 0$ the vortices in the «statistical gauge field» $a_\lambda(\boldsymbol{r}, t)$, of form $a_\lambda(\boldsymbol{r}, t) \sim (\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}})/er$, collect a local charge $n^{\text{loc}} = \nu$ around themselves (note that $A_\nu(\boldsymbol{r}, t)$ is the e.m. field).

Now it might be assumed that, because there is an energy gap $\Delta = \beta/\lambda$ in this theory, the charge is bound to the vortex cores in «sub-gap» states, as in superconducting vortices. But this is quite wrong. The eigenfunctions for (Ψ, Ψ^{+}) in the presence of a single vortex are easily found, and have the form (for $r^2 \gg l_{\text{H}}^2 = \hbar/eB$):

$$\begin{pmatrix} \Psi_{km}(\boldsymbol{r}) \\ \Psi^{+}_{km}(\boldsymbol{r}) \end{pmatrix} = \left(\frac{E_p - \Delta}{2E_p}\right)^{1/2} \begin{pmatrix} J_{|m-\nu|}(kr)\exp[im\phi] \\ J_{|m+1-\nu|}(kr)\exp[i(m+1)\phi] \end{pmatrix}, \quad (6)$$

where $\boldsymbol{p} = \hbar\boldsymbol{k}$, the quasi-particle energy $E_p \geq \Delta$, and $m = 0, \pm 1, \pm 2, \ldots$ (we assume $\nu < 1$). Then there are no bound states, for any T (if $\nu < 1$), and n^{loc} arises *entirely* from continuum states. The situation is the same as that prevailing in (2 + 1)-dimensional QED [12], and indeed we could not have a fractional n^{loc} if the states were bound!

It is then revealing to calculate n^{loc} around a vortex at finite T. A simple Boltzmann average then promotes charge higher up these states, and assuming $kT \ll \Delta$ (the Lagrangian (5) is unlikely to be meaningful otherwise), we find

$$n^{\text{loc}} \simeq \nu(1 - \exp[-\Delta/2kT]) \quad (7)$$

so that some charge has escaped (note that this result could also be obtained [1] by applying trace identities to (5)). In a real FQHE system there will be corrections to this arising from other quasi-particles or quasi-holes—these have long-range interactions. Nevertheless (7) clearly shows that the $T = 0$ Berry phase definition of n^{loc} will eventually fail at finite T (although if we had a finite-T microscope generalization of Laughlin's theory, presumably we could recover (7) as a Berry phase at finite T).

Experimental tests. – Let us briefly examine what is possible here. Recent experiments [13] have indicated how one may measure n^{loc} in the FQHE, and similar experiments should be capable of checking (7), thereby testing the effective action theories. A good way of testing the 3-d results in normal and superfluid ^{3}He would be via ballistic quasi-particles experiments involving thin wires [14], since these experiments see n^{loc} (not n_p^{glob}) for a quasi-particle «wave-packet». Similar experiments involving spin could be done by spin wave transmission (in metals or normal ^{3}He). In superconductors a convenient method would be to make a ballistic point contact spectroscopic measurement (using a polarized tip, if one is interested in S_p^{loc}). Detailed discussion of such experiments will be given elsewhere.

Thus, to conclude, we see that the «separation» of quasi-particle charges (*i.e.* the sometimes quite large differences between the local values of, *e.g.*, spin and fermionic charge) is a quite general phenomenon in both 2 and 3 dimensions—as is the distinction between the local and global values of each charge. This phenomenon arises because of interactions in 3 dimensions, at any temperature; and in 2 dimensions, even if there are topological terms in the effective action which may enforce quantized local charges at $T = 0$, these constraints break down at finite T. These local charges are often accessible experimentally, in both 2 and 3 dimensions.

* * *

I should like to thank Profs. G. SEMENOFF, W. KOHN and S. KIVELSON for discussions of these results. This work was supported by an NSERC-URF grant.

REFERENCES

[1] NIEMI A. J. and SEMENOFF G., *Phys. Rep.*, **135** (1986) 99.
[2] AFFLECK I., *J. Phys. Condensed Matter*, **1** (1989) 304.
[3] LAUGHLIN R. B., *The Quantum Hall Effect*, edited by R. E. PRANGE and S. M. GIRVIN (Springer, Berlin) 1987, p. 223.
[4] See, *e.g.*, CHEN Y. H. *et al.*, *Int. J. Mod. Phys. B*, **3** (1989) 1001 and references therein.
[5] BAYM G. and PETHICK C. J., *The Physics of Liquid and Solid Helium*, Vol. II (Wiley, New York, N.Y.) 1977, Chapt. 1.
[6] NOZIÈRES P., *Theory of Interacting Fermi Systems* (Benjamin, New York, N.Y.) 1964, Chapt. 6.
[7] HEINONEN O. and KOHN W., *Phys. Rev. B*, **36** (1987) 3565.
[8] KIVELSON S. A. and ROKHSAR D. S., ITP preprint (June 1989).
[9] The Yosida functions are defined in weak coupling in LEGGETT A. J., *Rev. Mod. Phys.*, **47** (1975) 331 and with strong-coupling corrections by SERENE J. W. and RAINER D., *Phys. Rep.*, **101** (1983) 221.
[10] LEINAAS J. M. and MYRHEIM J., *Nuovo Cimento B*, **37** (1977) 1. See also WU Y. S., *Phys. Rev. Lett.*, **52** (1984) 2103; **53** (1986) 111. Note that we are following standard terminology in talking about «fractional fermionic charge» (cf. ref. [1]) although it might be more consistent to talk about «anionic charge». The relation to Berry's phase in the FQHE is given by AROVAS D., SCHREIFFER J. R. and WILCZEK F., *Phys. Rev. Lett.*, **53** (1984) 722.

[11] GIRVIN S. M. and MACDONALD A. H., *Phys. Rev. Lett.*, **58** (1987) 1252; ZHANG S. C., HANSSON T. H. and KIVELSON S., *Phys. Rev. Lett.*, **62** (1989) 82; READ N., *Phys. Rev. Lett.*, **62** (1989) 86; these ideas have been generalized to the more complicated FQHE hierarchies by READ N., *Phys. Rev. Lett.*, **65** (1990) 1502.

[12] A similar phenomenon has been discussed in $(2+1)$-dimensional QED by DE VEGA H. J., *Phys. Rev. D*, **18** (1978) 2932 (at zero temperature only). Note that if $\nu > 1$, then there will be $[\nu]$ bound states at $T = 0$, plus $\nu - [\nu]$ local charge in continuum states.

[13] CLARK R. G. *et al.*, *Phys. Rev. Lett.*, **60** (1988) 1747; **62** (1989) 1536; SIMMONS J. A. *et al.*, *Phys. Rev. Lett.*, **63** (1989) 1731.

[14] See, *e.g.*, PICKETT G. R., *Rep. Prog. Phys.*, **51** (1988) 1295; LOUNASMAA O. V. and PICKETT G. R., *Sci. American*, **262**, No. 6 (June 1990), pp. 64-71.