

At the same time formula (2.2) is a superposition of plane waves propagating throughout the crystal. The wave vector  $\mathbf{k}$  has the same properties as  $\mathbf{p}/\hbar$  (where  $\mathbf{p}$  is the quasimomentum). The index  $s$  denotes the type of wave, and the unit vector of polarization  $\mathbf{e}_j$  defines how various atoms oscillate in a single unit cell. If the unit cell contains  $z$  atoms, then the index  $s$  takes on  $3z$  various values. The vibration frequency  $\omega$  depends on  $\mathbf{k}$  and  $s$ .

Formula (2.2) is reminiscent of the wave function of free particles:

$$\exp\left(i\frac{\mathbf{p}\mathbf{r}}{\hbar} - i\frac{\varepsilon t}{\hbar}\right).$$

The role of the momentum  $\mathbf{p}$  is played by  $\hbar\mathbf{k}$  and that of the energy by  $\hbar\omega$ . Using this, we can introduce a new physical picture. Usually we deal with real particles, whose free motion is described by a plane wave. In this particular case we will treat expression (2.2) as the wave function of certain fictitious particles, which we call "quasiparticles". Since this notion is universal, the quasiparticles that correspond to lattice vibrations are specifically called "phonons". The origin of this term is associated with the fact that the quasiparticles in question have the same relation to elastic waves propagating in the lattice (i.e., to sound) as light quanta have to electromagnetic waves. Thus, it may be said briefly that usually in quantum mechanics waves describe the motion of particles and here particles are introduced for the description of waves.

The meaning of the description by means of quasiparticles becomes clearer if we consider the energy of a vibrating crystal. The energy levels are expressed by the formula for a system of noninteracting oscillators:

$$E - U_0 = \sum_{\mathbf{k}, s} \hbar\omega(\mathbf{k}, s) \left(n(\mathbf{k}, s) + \frac{1}{2}\right). \quad (2.3)$$

The numbers  $n$  are either equal to zero or positive integers. Let us write expression (2.3) as the sum of two terms:

$$E - U_0 = \frac{1}{2} \sum_{\mathbf{k}, s} \hbar\omega(\mathbf{k}, s) + \sum_{\mathbf{k}, s} \hbar\omega(\mathbf{k}, s) n(\mathbf{k}, s). \quad (2.4)$$

The first term corresponds to the lowest value of the energy and describes the ground state of the system. This is the energy of the so-called zero-point vibrations. The fact that the atoms of the crystal lattice must be vibrating even in the ground state is associated with the quantum uncertainty principle. According to this principle, a particle cannot be at rest in the equilibrium position, since in such a case it would have simultaneously a certain coordinate and a certain momentum.

In an excited state the numbers  $n(\mathbf{k}, s)$  are different from zero. Formula (2.4) corresponds in this case to a system of independent particles with energies  $\hbar\omega(\mathbf{k}, s)$ . Since the numbers  $n(\mathbf{k}, s)$  can take on any positive integer values, it follows that any number of phonons may be in the same state. This means that they obey Bose statistics.

The concept of phonons is valid as long as the vibrational amplitude is small compared to the lattice period. Otherwise, one has to take into account the terms in the expansion of the potential energy  $U$  in higher powers of the displacement, and the total energy can no longer be expressed by formula (2.3). However, this occurs only near the melting point.

According to the idea put forward by Landau, any homogeneous system composed of a large number of particles has low-lying excited states of the same type as the vibrating lattice. Namely, the properties of any system may be described in terms of the quasiparticle model. Quasiparticles may have either an integer ( $n\hbar$ ) or a half-integer ( $(n + \frac{1}{2})\hbar$ ) spin, i.e., they may be either Bose particles or Fermi particles. The statistics of quasiparticles is not related uniquely to the statistics of the particles that make up the system. For example, as we have seen, phonons always obey Bose statistics, irrespective of the spin of the atoms that make up the lattice. The energy of quasiparticles is a function of their momentum. This dependence  $\varepsilon(p)$  is the main characteristic of the low-lying excited states.

## 2.2. *Quasiparticles in an isotropic Fermi liquid*

Electrons have spin  $\frac{1}{2}\hbar$ . In view of this, the electron liquid is a so-called Fermi liquid. What are the properties of quasiparticles in such a liquid? According to the Landau hypothesis (1956), the energy spectrum of such a liquid is very similar to the spectrum of an ideal Fermi gas. The validity of this hypothesis was later rigorously proved. We do not give this proof here because it by far exceeds in complexity the level of this book\*.

So, we begin with the ideal gas. The equilibrium distribution function is the well-known Fermi function\*\*:

$$f = [e^{(\varepsilon - \mu)/T} + 1]^{-1}$$

Here  $\varepsilon = p^2/2m$  and  $\mu$  is the chemical potential. At  $T = 0$  we have  $f = 1$  if  $\varepsilon < \mu(0)$ , and  $f = 0$  if  $\varepsilon > \mu(0)$  (fig. 6, solid line). The quantity  $\mu(0)$  is called the Fermi level. Introducing the Fermi momentum  $p_0$  in accordance with the formula  $\mu(0) = p_0^2/2m$ , we find that at  $T = 0$  all the states contained in a sphere of radius  $p = p_0$  (the Fermi sphere) in momentum space are occupied, all the external states being free. This is a consequence of the Pauli exclusion principle - only one particle may be in each of the states, and in this particular case at  $T = 0$  the lower states are occupied. The occupied volume in the phase space of momenta, coordinates and spins divided by  $(2\pi\hbar)^3$  must be equal to the number of particles. The volume in momentum space

\* The proof can be found in the book by Abrikosov et al. (1962).

\*\* Here and henceforth the temperature is determined in energy units. To convert to degrees, it must be divided by the Boltzmann constant  $k_B = 1.38 \times 10^{-16}$  erg/K.

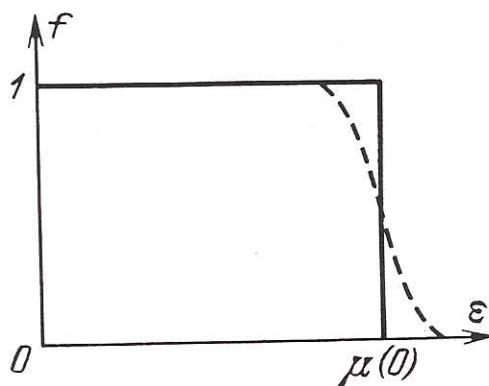


Fig. 6.

is the volume of the Fermi sphere. The possibility of the occurrence of two values for the spin projection is given by the factor 2. In view of this, we obtain:

$$N = 2\left(\frac{4}{3}\pi\right)p_0^3 V (2\pi\hbar)^{-3} \quad \text{or} \quad p_0 = \hbar(3\pi^2 N/V)^{1/3}. \quad (2.5)$$

We now turn to  $T \neq 0$ . The distribution function in this case is given by the dashed curve in fig. 6. The width of the "smeared-out" region is of the order of  $T$ . This is associated with the fact that some of the particles, after having received an extra energy of order  $T$ , escape from the Fermi sphere. The equilibrium state at  $T \neq 0$  and, in general, any excited state can be generated from the state at  $T = 0$  by way of successive displacements of particles from the interior of the Fermi sphere to the outside. Each such act results in a particle outside the Fermi sphere and a free site or an "antiparticle" inside it. These particles and antiparticles represent in this case the quasiparticles of the excited state\*. Their energy must be counted from the Fermi level  $\mu(0)$ . Quasiparticles of particle-type have momenta larger than  $p_0$  and their energy is given by

$$\xi_p(p) = \frac{p^2}{2m} - \frac{p_0^2}{2m}.$$

If  $p - p_0 \ll p_0$ , then

$$\xi_p \approx v(p - p_0),$$

where  $v_0 = p_0/m$  is the velocity at the Fermi sphere. On the other hand, quasiparticles of antiparticle-type have momenta smaller than  $p_0$  and their energy must be counted in the opposite direction:

$$\xi_a(p) = \frac{p_0^2}{2m} - \frac{p^2}{2m}$$

\* The "antiparticles" constitute a full analogy to anti-particles in the theory of elementary particles (for example, the positron). The term "holes" frequently used for such quasiparticles is unjustified in our opinion because this term is used to denote another object: vacant sites in an unfilled band (Section 2.3).

or

$$\xi_a(p) \approx v(p_0 - p)$$

if  $p_0 - p \ll p_0$ . Such counting-off of the energy is due to the fact that the creation of antiparticles in the depth of the Fermi distribution requires a larger consumption of energy than at the Fermi surface.

According to the Landau hypothesis, the spectrum of quasiparticles in an isotropic Fermi liquid with a strong interaction between the particles is constructed in the same way as for the ideal gas. This means that there exists a certain value of  $p_0$ , which is connected, in accordance with the Landau theory, with the density of particles by the same relation as in the case of the ideal gas (formula 2.5). There are two types of quasiparticles: "particles" with  $p > p_0$  and "antiparticles" with  $p < p_0$ . Their energies for the case  $|p - p_0| \ll p_0$  are, respectively, equal to

$$\xi_p \approx v(p - p_0), \quad \xi_a \approx v(p_0 - p). \quad (2.6)$$

However, in this case  $v$  is simply a certain unknown coefficient, which has the dimension of velocity. Instead of  $v$  we may introduce another coefficient with the aid of the following relation:

$$v = p_0 / m^*. \quad (2.7)$$

The constant  $m^*$  with the dimension of mass is called the effective mass\*.

As has been pointed out above, these assumptions of the spectrum have been verified in a rigorous but rather complicated manner, but we can offer a simpler reasoning. If the state corresponding to the presence of a quasiparticle is not a true stationary state of the Fermi liquid, it must attenuate with time due to transitions to other states. The corresponding wave function will thus have the form

$$\exp\left[-i \frac{\xi(p)t}{\hbar} - \frac{\gamma(p)t}{\hbar}\right]. \quad (2.8)$$

We may meaningfully speak of quasiparticles only in those cases when  $\gamma \ll |\xi|$ . We have therefore to estimate  $\gamma$ . Evidently, it is proportional to the probability of the transition of the state under consideration to other states.

Let us first define this probability for a weakly interacting gas. If there is a particle 1 outside the Fermi distribution, then the process of first order in the interaction will be as follows (fig. 7). Particle 1 interacts with particle 2 inside the Fermi sphere, following which the two particles pass over to states 1' and 2' outside the Fermi sphere. Because of the Pauli principle this is the only possibility. The law of momentum conservation requires that

$$p_1 + p_2 = p'_1 + p'_2$$

\* In various books and articles the term "effective mass" is used for rather diverse quantities with the dimension of the mass. We shall use this term only for the isotropic spectrum.

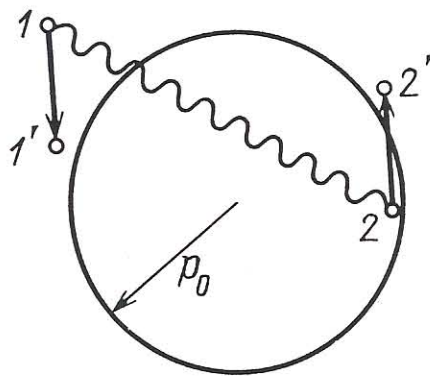


Fig. 7.

and, in accordance with what has been said above,

$$p_1 > p_0, \quad p_2 < p_0, \quad p_1' > p_0, \quad p_2' > p_0.$$

The momentum conservation is shown graphically in fig. 8. The planes  $(p_1, p_2)$  and  $(p_1', p_2')$  do not coincide, generally speaking, and in fig. 8 they are simply superposed by rotation.

The scattering probability is given to within a constant by

$$W \propto \int \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2') d^3 p_2 d^3 p_1'.$$

The integration is carried out only over  $p_2$  and  $p_1'$ , since  $p_2'$  is determined by the law of momentum conservation. The angle between the vectors  $p_1'$  and  $p_2'$  is actually specified by the law of energy conservation. The integration over this angle eliminates the  $\delta$ -function. It now remains only to integrate over the absolute values of the vectors.

Suppose that  $p_1$  is close to  $p_0$ . Then, all the remaining momenta will also be close to  $p_0$  in absolute value, and, consequently, in fig. 8 they will make nearly equal angles with the horizontal line (with the sum  $p_1 + p_2$ ). Hence, from the relationship between the projections on this axis we can write the relation between the absolute

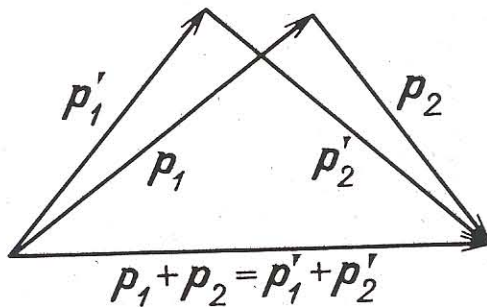


Fig. 8

values:  $p'_1 \approx p_1 + p_2 - p'_2$ . Since  $p'_2 > p_0$ , it follows that we have  $p'_1 < p_1 + p_2 - p_0$ . But at the same time  $p'_1 > p_0$ , from which it follows that  $p_1 + p_2 - p_0 > p_0$  or  $p_2 > 2p_0 - p_1$ . The upper limit for  $p_2$  is  $p_0$ . Thus, we have

$$0 > p_2 - p_0 > p_0 - p_1, \quad 0 < p'_1 - p_0 < (p_1 - p_0) + (p_2 - p_0).$$

On integration we obtain

$$\int dp_2 dp'_1 = \frac{1}{2}(p_1 - p_0)^2.$$

Hence,

$$\gamma \propto W \propto (p_1 - p_0)^2. \quad (2.9)$$

The complete formula for  $\gamma$  can be obtained from dimensionality considerations. It must be proportional to the square of the interaction constant and, according to the above calculation, to the quantity  $(p - p_0)^2$ . Following this, we have to introduce a further factor made up of  $p_0$ ,  $m$  and  $\hbar$  in such a manner that the result has the dimension of energy.

Let us now turn to a strongly interacting system. As the interaction constant increases, other processes involving a larger number of particles may, in principle, become important, but it can be shown that the probabilities of such processes will contain higher powers of  $p - p_0$ . Hence, at  $|p - p_0| \ll p_0$  the process in question will nevertheless predominate, i.e., we will again have  $\gamma \propto (p - p_0)^2$ . As for the other factors, we have to take into account the fact that in the liquid, whose volume is determined by the forces of particle interaction rather than by the walls of the container, the density is always such that the average kinetic energy of the particles and their potential energy of interaction will be approximately equal. It means that there is only one energy scale - the Fermi energy  $\mu(0)$  or  $p_0 v$ . From this it follows that the quantity with the dimension of energy, which is proportional to  $(p - p_0)^2$ , must be equal to

$$\gamma = \alpha \frac{\xi^2}{\mu} \quad (2.10)$$

where  $\alpha \sim 1$ .

As has been said above, the quasiparticle concept is valid at  $\gamma \ll |\xi|$ . This really occurs near the Fermi level, i.e., at  $|\xi| \ll \mu$ . This justifies the above assumption of the spectrum in the vicinity of the Fermi level, i.e., for quasiparticles with small energies  $\xi$ .

If we deal with an equilibrium Fermi liquid at  $T \neq 0$ , the quasiparticles in it always have an energy  $\xi \approx T$ . The attenuation  $\gamma$  will be of order  $T^2/\mu$ . It follows from this that the description of the liquid in terms of quasiparticles will be valid only as long as  $T \ll \mu$ .

The quantity  $\mu$  can be estimated using the gas model. For electrons in a metal the value of  $\hbar/p_0$  (the de Broglie wavelength) is of the order of interatomic spacings,

i.e.,  $10^{-8}$  cm and, hence,  $p_0 \sim 10^{-19}$  g · cm/s. It then follows that  $p_0^2/2m \sim 1-10$  eV or, dividing by Boltzmann's constant, we find

$$T \ll T_0 \sim 10^4-10^5 \text{ K.} \quad (2.11)$$

This condition shows that the quasiparticle picture can really be applied to solid metals at all temperatures because  $T_0$  is appreciably higher than the melting point in all cases.

Formulas (2.6) for the energy spectrum of quasiparticles may be written in a unified manner in the following form:

$$\varepsilon_{qp} = |\xi|, \quad \xi = \varepsilon(\mathbf{p}) - \mu. \quad (2.6')$$

Here it must be kept in mind that the "antiparticles" have a charge which is opposite to the charge of the "particles". However, we may introduce another, more familiar object. Let us visualize an ideal Fermi gas with a density  $N/V$ , which is composed of particles with a mass  $m^*$ . The spectrum of quasi-particles of such a gas is the same as in the case of the Fermi liquid. Therefore, such an ideal gas may describe the properties of a real interacting system. However, one has to bear in mind that the properties of the gas model that depend on the particles located far from the Fermi level do not correspond to a real Fermi liquid. In what follows, depending on the convenience, we will make use of both pictures: the gas model or quasiparticles with the spectrum (2.6').

### 2.3. The anisotropic Fermi liquid

All the results described above refer to an isotropic Fermi liquid. In order to ascertain the meaning of the electronic spectra of metals, we first "turn off" the interaction of electrons or, more precisely, we will consider a gas composed of noninteracting electrons placed in an averaged periodic field. The states of one particle in such a field have been considered in the preceding chapter. It has been shown that the energy levels form bands separated by forbidden portions (energy gaps). Each band has  $2N$  states, where  $N$  is the number of unit cells in the specimen.

If there are many noninteracting particles, they are distributed in some way between these states. At  $T=0$  (and in metals practically at all temperatures below the melting point) all the lower states will be occupied up to a certain maximum level (the Fermi energy) and all the higher states will be empty. There are two possibilities here.

(1) The Fermi level coincides with the upper edge of one of the bands, so that some of the bands are completely filled and the others are quite empty. In this case, not too strong an electric field cannot produce electric current. This follows from the fact that in the equilibrium state to each electron with momentum  $\mathbf{p}$  there corresponds another one with momentum  $-\mathbf{p}$  since  $\varepsilon(\mathbf{p})$  is an even function of the momentum. Therefore, there is no current in the equilibrium state. In order to produce current, it is necessary to redistribute the electrons between these states.

## *Fermi-liquid effects*

### *13.1. Interaction of quasiparticles*

In ch. 2, while speaking of the correspondence between a liquid consisting of Fermi particles (a Fermi liquid) and an ideal Fermi gas, we deliberately omitted one important point. According to the Landau theory, there is one important distinction between the spectra of the quasiparticles of the Fermi liquid and those of the quasiparticles of the Fermi gas. While in the case of the Fermi gas the shape of the energy spectrum (2.6) is determined by the energy of a free particle alone, in the Fermi liquid a significant role is played by the interaction with other quasiparticles, which is not weak, generally speaking.

In ch. 2 we have found out that that this interaction does not lead to a strong damping if the energy of the quasiparticle is close to the Fermi energy. But this was associated not with the weakness of the interaction but with the form of the Fermi spectrum (the presence of an occupied Fermi sphere). The fact that the interaction is not weak manifests itself, in particular, in that the effective mass of quasiparticles may be appreciably different from the mass of free particles (for example, in liquid  $^3\text{He}$  at a low temperature  $m^* = 3m_{\text{He}}$ ; in metals this difference is smaller: see ch. 14).

According to the idea expressed by Landau (1956), the interaction of quasiparticles may be introduced as a certain self-consistent field generated by surrounding quasiparticles, which acts on a given quasiparticle. But here the energy of the quasiparticle will evidently depend on the state of other quasiparticles, i.e., in other words, it will be a functional of their distribution function\*.

In such a case, the following questions arise. First, how can this energy be determined? Second, what is the equilibrium distribution function? Third, how can the nonequilibrium function be found? In other words, strictly speaking, one has to reconsider all the derivations and results obtained before, since they have, in fact, used the gas model without taking account of the dependence of the energy spectrum on the distribution function.

As will be shown in this chapter, this new circumstance, in effect, does not prove to be important, with rare exceptions. In some cases it leads to numerical differences

\* In this chapter, the term "quasiparticles" is used to describe particles of the gas model of the Fermi liquid. But, in contrast to the preceding chapters, the interaction of quasiparticles will be taken into account.



but does not alter the order of magnitude of the result. But there are phenomena which occur exclusively due to the dependence of the spectrum on the distribution function. Such phenomena are known as Fermi-liquid effects. For the sake of simplicity, we shall consider the isotropic model in the following.

In this chapter we shall use a different notation for the distribution function, namely  $n(\mathbf{p}, \mathbf{r})$ . This is because the letter  $f$  in the theory of the Fermi liquid always denotes the Landau function (see below) and the use of another symbol for the latter seems to be unreasonable.

We begin with the energy of quasiparticles. Let the total energy of the system be  $E[n]$ , where  $[n]$  signifies the functional dependence on the distribution function. It is natural to define the energy of quasiparticles as follows.

With a small change in the distribution function  $\delta n$  the variation of the total energy per unit volume is given by

$$\delta E = \sum_{\sigma} \int \varepsilon(\mathbf{p}, \sigma) \delta n(\mathbf{p}, \sigma) \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3}, \quad (13.1)$$

where  $\sum_{\sigma}$  denotes the summation over spin projections. The quantity  $\varepsilon(\mathbf{p}, \sigma)$  is naturally regarded as the energy of quasiparticles. Indeed, if one quasiparticle of momentum  $p_1$  and spin projection  $\sigma_1$  appears, then  $\delta n(\mathbf{p}, \sigma) = (2\pi\hbar)^3 \delta(\mathbf{p} - \mathbf{p}_1) \delta_{\sigma\sigma_1}$ . But in this case  $\delta E = \varepsilon(\mathbf{p}_1, \sigma_1)$ .

Generally speaking, along with the momentum and spin, the coordinate should be introduced here. However, actually, because of the wavelength of electrons being of the order of interatomic distances, the inhomogeneities of the distribution function under consideration, with characteristic dimensions much larger than the interatomic distances, will not affect the energy of quasiparticles.

It is convenient to write formula (13.1) in a different form. If the spin of the electron plays a significant role in the problem, then, instead of this distribution function, one has to employ a so-called statistical operator or a density matrix:  $n(\mathbf{p}, \sigma)$ . Here, the average value of any quantity, to which the operator  $A$  dependent on the spin and momentum operators corresponds, is expressed by the formula

$$\bar{A} = \text{Sp}_{\sigma} \int A n(\mathbf{p}, \sigma) \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3}. \quad (13.2)$$

In particular, formula (13.1) may also be written in this form:

$$\delta E = \text{Sp}_{\sigma} \int \varepsilon(\mathbf{p}, \sigma) \delta n(\mathbf{p}, \sigma) \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3}. \quad (13.3)$$

The determination of the energy of quasiparticles in accordance with (13.3), as we shall prove, gives a Fermi distribution in the case of equilibrium. To this end, we write the formula for the entropy:

$$S = -\text{Sp}_{\sigma} \int \{n \ln n + (1-n) \ln(1-n)\} \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3}. \quad (13.4)$$

The validity of this formula as applied to quasiparticles in a Fermi liquid is proved by the fact that it is of purely combinatorial origin, and for quasiparticles the Pauli principle is valid just as for particles of a Fermi gas. Let us now find the maximum of the entropy, requiring that the number of particles and the total energy be constant. This can be done by seeking the maximum of the expression

$$S' = S + \alpha E + \beta N,$$

where  $\alpha$  and  $\beta$  are undetermined Lagrange multipliers. By varying this expression with respect to  $n$ , from the condition  $\delta S' = 0$  we find the equilibrium distribution function:

$$n_0(\varepsilon) = n_F(\varepsilon) = [e^{(\varepsilon - \mu)/T} + 1]^{-1}, \quad (13.5)$$

where  $\mu$  and  $T$  are expressed in terms of  $\alpha$  and  $\beta$ .

Formulae (13.5) and (13.3) immediately prove the correctness of the definition of heat capacity in section 2.4. Formula (2.18) corresponds only formally to (13.3). In fact, there is a distinction between these formulas, since eq. (2.18) ignores the variation of  $\varepsilon$  upon variation of the distribution function with temperature. However, this gives only small corrections to the heat capacity, of the order of  $(T/\mu)^2$ .

### 13.2. The Landau function

Although the dependence of the spectrum on the distribution function does not manifest itself in the heat capacity, it nevertheless does occur. If  $E$  is a functional of  $n$ , then this is also valid for  $\varepsilon$ . With a slight variation of the distribution function the function  $\varepsilon$  acquires a correction, which may be written in the form

$$\delta\varepsilon(\mathbf{p}, \boldsymbol{\sigma}) = \text{Sp}_{\boldsymbol{\sigma}'} \int f(\mathbf{p}, \boldsymbol{\sigma}; \mathbf{p}', \boldsymbol{\sigma}') \delta n(\mathbf{p}', \boldsymbol{\sigma}') \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}. \quad (13.6)$$

Since the function  $f$  is a second variational derivative of the total energy, it is symmetrical with respect to the transpose of the arguments, i.e.,

$$f(\mathbf{p}, \boldsymbol{\sigma}; \mathbf{p}', \boldsymbol{\sigma}') = f(\mathbf{p}', \boldsymbol{\sigma}'; \mathbf{p}, \boldsymbol{\sigma}).$$

The existence of relation (13.6) manifests itself, first of all, in the kinetic equation. In section 3.2 in deriving the kinetic equation we had

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \left(\frac{\partial n}{\partial \mathbf{r}}\right) \frac{d\mathbf{r}}{dt} + \left(\frac{\partial n}{\partial \mathbf{p}}\right) \frac{d\mathbf{p}}{dt}.$$

The quantity  $d\mathbf{p}/dt$  was determined by the acting force. It was presumed that the only force that arises comes from the external field. However, in the case under consideration, due to relation (13.6), the energy of quasiparticles begins to depend on the coordinate via the distribution function  $n$ . This implies the appearance of a

potential energy, which describes the action on a given quasiparticle of the self-consistent field of the other quasiparticles. Hence, we must now write

$$\begin{aligned} \frac{dp}{dt} &= eE - \frac{\partial \varepsilon}{\partial \mathbf{r}} \\ &= eE - \text{Sp}_{\sigma'} \int f(\mathbf{p}, \boldsymbol{\sigma}; \mathbf{p}', \boldsymbol{\sigma}') \frac{\partial n}{\partial \mathbf{r}} \frac{d^3 \mathbf{p}'}{(2\pi\hbar)^3}. \end{aligned}$$

Thus, an additional term appears on the left-hand side of the kinetic equation. If we assume that  $n = n_0(\varepsilon_0) + n_1$ , where  $\varepsilon_0$  is the energy at equilibrium at  $T=0$ , and retain only terms linear in  $n_1$ , then the additional term has the form

$$-\frac{\partial n_0}{\partial \varepsilon} \mathbf{v} \text{Sp}_{\sigma'} \int f(\mathbf{p}, \boldsymbol{\sigma}; \mathbf{p}', \boldsymbol{\sigma}') \frac{\partial n_1}{\partial \mathbf{r}} \frac{d^3 \mathbf{p}'}{(2\pi\hbar)^3}.$$

Finally, if we seek  $n_1$  in the form  $n_1 = -\psi \partial n_0 / \partial \varepsilon$ , then by dividing the entire equation by  $\partial n_0 / \partial \varepsilon$  we obtain the additional term in the equation for  $\psi$  in the form

$$-\frac{\mathbf{v}}{(2\pi\hbar)^3} \text{Sp}_{\sigma'} \int f(\mathbf{p}, \boldsymbol{\sigma}; \mathbf{p}', \boldsymbol{\sigma}') \frac{\partial}{\partial \mathbf{r}} \psi(\mathbf{r}, \mathbf{p}', \boldsymbol{\sigma}') \frac{dS'}{v},$$

where the momenta  $\mathbf{p}$  and  $\mathbf{p}'$  are on the Fermi surface.

Thus, the kinetic equation may be written in the form

$$\begin{aligned} \frac{\partial \psi}{\partial t} + \mathbf{v} \frac{\partial}{\partial \mathbf{r}} \left( \psi + \text{Sp}_{\sigma'} \int f(\mathbf{p}, \boldsymbol{\sigma}; \mathbf{p}', \boldsymbol{\sigma}') \psi(\mathbf{p}', \boldsymbol{\sigma}') \frac{dS'}{v(2\pi\hbar)^3} \right) \\ - e\mathbf{v}E = I(\psi). \end{aligned} \tag{13.7}$$

In the presence of not too strong a magnetic field one more term appears on the left-hand side of the kinetic equation:

$$\frac{e}{c} \left[ \frac{\partial \varepsilon}{\partial \mathbf{p}} \mathbf{H} \right] \frac{\partial n}{\partial \mathbf{p}}.$$

In the zeroth approximation this gives zero since  $n_0$  depends only on  $\varepsilon$ . In the first approximation one has to take into account the term with  $\psi$  not only in  $\partial n / \partial \mathbf{p}$  but also in  $\partial \varepsilon / \partial \mathbf{p}$ . A simple calculation shows that this leads to the appearance in eq. (13.7) of the following term:

$$\frac{e}{c} [\mathbf{vH}] \frac{\partial}{\partial \mathbf{p}} \left( \psi + \text{Sp}_{\sigma'} \int f(\mathbf{p}, \boldsymbol{\sigma}; \mathbf{p}', \boldsymbol{\sigma}') \psi(\mathbf{p}', \boldsymbol{\sigma}') \frac{dS'}{v(2\pi\hbar)^3} \right).$$

The operator  $(e/c)[\mathbf{vH}]\partial/\partial \mathbf{p}$  is nothing else than  $\partial/\partial t_1$ , where  $t_1$  is the variable introduced in section 5.1.

Consider the collision integral. It is obvious that it vanishes only if  $n_0(\varepsilon)$  with a real energy  $\varepsilon$  rather than  $n_0(\varepsilon_0)$  is substituted into it. We may write

$$n = n_0(\varepsilon_0) + n_1 = n_0(\varepsilon) - \frac{\partial n_0}{\partial \varepsilon} \text{Sp}_{\sigma'} \int f n_1 \frac{d^3 p'}{(2\pi\hbar)^3} + n_1.$$

Substituting here  $n_1 = -\psi \partial n_0 / \partial \varepsilon$ , we obtain

$$n = n_0(\varepsilon) - \frac{\partial n_0}{\partial \varepsilon} \left[ \psi + \text{Sp}_{\sigma'} \int f \psi \frac{dS'}{v(2\pi\hbar)^3} \right].$$

Since the collision integral vanishes upon substitution of  $n_0(\varepsilon)$ , it evidently follows that it will depend on the function

$$\varphi = \psi + \text{Sp}_{\sigma'} \int \frac{f \psi dS'}{v(2\pi\hbar)^3} \quad (13.8)$$

in the same way as it was dependent on  $\psi$  at  $f=0$ .

Let us now find the expression for the electric current in terms of  $\psi$ :

$$\begin{aligned} \mathbf{j} &= e \text{Sp}_{\sigma} \int \frac{\partial \varepsilon}{\partial \mathbf{p}} n \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3} \\ &= e \text{Sp}_{\sigma} \int v n_1 \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3} + e \text{Sp}_{\sigma, \sigma'} \int n_0 \frac{\partial}{\partial \mathbf{p}} \int f n_1 \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3} \frac{d^3 \mathbf{p}'}{(2\pi\hbar)^3} \end{aligned}$$

( $v = \partial \varepsilon_0 / \partial \mathbf{p}$ ). Taking the integral over  $\mathbf{p}$  by parts, we have

$$\mathbf{j} = e \text{Sp}_{\sigma} \int v n_1 \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3} - e \text{Sp}_{\sigma, \sigma'} \int \frac{\partial n_0}{\partial \varepsilon} v \int f n_1 \frac{d^3 \mathbf{p}}{(2\pi\hbar)^3} \frac{d^3 \mathbf{p}'}{(2\pi\hbar)^3}.$$

Finally, substituting  $n_1 = -\psi \partial n_0 / \partial \varepsilon$ , we have

$$\mathbf{j} = e \text{Sp}_{\sigma} \int \left[ \psi + \text{Sp}_{\sigma'} \int f \psi \frac{dS'}{v(2\pi\hbar)^3} \right] v \frac{dS}{v(2\pi\hbar)^3}. \quad (13.9)$$

Thus, it turns out that with the  $f$ -function taken into account the expression for the current and the collision integral depend on the combination of  $\varphi$  in (13.8) in the same way as they were dependent on  $\psi$  at  $f=0$ . It can easily be shown that this is valid for the heat flux as well. If  $\psi$  is independent of the time or  $\omega$  is much smaller than the characteristic frequency for the problem under consideration, then the term  $\partial \psi / \partial t$  is absent on the left-hand side of the kinetic equation (13.7). But in this case too, it contains the same combination of  $\varphi$  (13.8). It thus follows that in all the phenomena considered earlier, for which it may be assumed that  $\omega = 0$  in the kinetic equation, the presence of the  $f$ -function plays no role whatsoever.

The function  $f$  plays a role only in those kinetic phenomena for which the term  $\partial \psi / \partial t$  in the kinetic equation is important. However, even in this case it does not always occur. Consider, for example, the anomalous skin effect or cyclotron resonance (see ch. 7). We have found that the function  $\psi$  is almost  $\delta$ -shaped:  $\psi \propto \delta(v_x)$ ,

where  $v_x$  is the velocity component normal to the surface. Of course, the function  $\psi$  has also a smooth part, but in the problems under discussion it is not important. It is not difficult to see that upon substitution of the  $\delta$ -function into the integral (13.8) we obtain a smooth function. Hence, the  $\delta$ -shaped parts of the functions  $\psi$  and  $\varphi$  are identical. Thus, the difference between  $\psi$  and  $\varphi$  may be ignored, which completely justifies the calculations performed with the function  $f$  being neglected.

Finally, in some of the problems discussed earlier in this book the term  $\partial\psi/\partial t$  is essential and at the same time the  $\psi$ -function does not exhibit a  $\delta$ -shaped character. For example, this refers to the problem of the propagation of magnetoplasmon waves in an even metal. In this case, the term  $v\partial\psi/\partial r$  is not essential, but the term  $\partial\psi/\partial t$  is present. The difference from the case with  $f=0$  consists in the following: in the current and in the collision integral  $\psi$  is replaced by  $\varphi = L\psi$ , where  $L$  is a real linear integral operator. By order of magnitude  $L \sim 1$ .

From the foregoing it follows that the statement that at  $\omega\tau \gg 1$  the kinetic coefficients will depend on  $-i\omega$  in the same manner as they were dependent on  $\tau^{-1}$  at  $\omega\tau \ll 1$  is still correct in order of magnitude and with respect to the complex nature of the coefficients. But the quantity  $\tau$  itself was determined only by order of magnitude. Hence, the presence of the function  $f$  in this case does not lead to a qualitative change in the results. However, in concrete calculations with a given form of the spectrum the function  $f$  must be taken into account.

Thus, practically in all calculations where we can apply the concept of the quasiclassical electron moving along an orbit, the Fermi-liquid effects do not practically manifest themselves. It is for this reason that in order to avoid complications in calculations and in their interpretation we have not introduced the  $f$ -function in the preceding chapters.

### 13.3. The role of the interaction of quasiparticles in paramagnetic susceptibility

In section 10.1 we have considered the simplest quantum phenomenon - spin paramagnetism. According to formula (10.2), the paramagnetic susceptibility  $\chi$  contains a single characteristic of a given metal: the density of states  $\nu(\mu)$ , i.e., the same quantity which, according to formula (2.28), determines the electronic heat capacity. As has been noted in section 3.1, the formula for the heat capacity is not changed when account is taken of quasiparticle interaction. However, as we will show below, the function  $f$ , while not altering the order of magnitude of  $\chi$ , leads to a change in its particular value\*.

\* In metals the exact value of  $\chi$  is not of great interest since only the total susceptibility can be measured, and the calculation of the diamagnetic part presents serious difficulties. But there exists a neutral Fermi liquid - liquid  $^3\text{He}$  - which exhibits only spin paramagnetism. Then, by comparing  $\chi$  with the heat capacity one can extract information about the function  $f$ .