Fractional Statistics and the Quantum Hall Effect

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The statistics of quasiparticles entering the quantum Hall effect are deduced from the adiabatic theorem. These excitations are found to obey fractional statistics, a result closely related to their fractional charge.

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Extensive experimental studies have been carried out[1] on semiconducting heterostructures in the quantum limit $\omega_0\tau >> 1$, where $\omega_0 = eB_0/m$ is the cyclotron frequency and τ is the electronic scattering time. It is found that as the chemical potential μ is varied, the Hall conductance $\sigma_{xy} = I_x/E_y = \nu e^2/h$ shows plateaus at $\nu = n/m$, where n and m are integers with m being odd. The ground state and excitations of a two-dimensional electron gas in a strong magnetic field B_0 have been studied[2–4] in relation to these experiments and it has been found that the free energy shows cusps at filling factors $\nu = n/m$ of the Landau levels. These cusps correspond to the existence of an "incompressible quantum fluid" for given n/m and an energy gap for adding quasiparticles which form an interpenetrating fluid. This quasiparticle fluid in turn condenses to make a new incompressible fluid at the next larger value of n/m, etc.

The charge of the quasiparticles was discussed by Laughlin[2] by using an argument analogous to that used in deducing the fractional charge of solitons in one-dimensional conductors.[5] He concluded for $\nu = 1/m$ that quasiholes and quasiparticles have charges $\pm e^* = \pm e/m$. For example, a quasihole is formed in the incompressible fluid by a two-dimensional bubble of a size such that $1/m$ of an electron is removed. Less clear, however, is the statistics which the quasiparticles satisfy; Fermi, Bose, and fractional statistics having all been proposed. In this Letter, we give a direct method for determining the charge and statistics of the quasiparticles.

In the symmetric gauge $\vec{A}(\vec{r}) = \frac{1}{2}\vec{B}_0 \times \vec{r}$ we consider the Laughlin ground state with filling factor $\nu = 1/m$,

$$\psi_m = \prod_{j<k} (z_j - z_k)^m \exp(-\tfrac{1}{4}\textstyle\sum_l |z_l|^2), \tag{1}$$

where $z_j = x_j + iy_j$. A state having a quasihole localized at z_0 is given by

$$\psi_m^{+z_0} = N_+ \textstyle\prod_i (z_i - z_0)\psi_m, \tag{2}$$

while a quasiparticle at z_0 is described by

$$\psi_m^{-z_0} = N_- \textstyle\prod_i (\partial/\partial z_i - z_0/a_0^2)\psi_m, \tag{3}$$

where $2\pi a_0^2 B_0 = \phi_0 = hc/e$ is the flux quantum and $N_\pm$ are normalizing factors.

To determine the quasiparticle charge e^*, we calculate the change of phase γ of $\psi_m^{+z_0}$ as z_0 adiabatically moves around a circle of radius R enclosing flux ϕ. To determine e^*, γ is set equal to the change of phase,

$$(e^*/\hbar c)\oint \vec{A}\cdot d\,\vec{l} = 2\pi(e^*/e)\phi/\phi_0, \tag{4}$$

that a quasiparticle of charge e^* would gain in moving around this loop. As emphasized recently by Berry[6] and by Simon[7] (see also Wilczek and Zee[8] and Schiff[9]), given a Hamiltonian $H(z_0)$ which depends on a parameter z_0, if z_0 slowly transverses a loop, then in addition to the usual phase $\int^t E(t')\,dt'$, where $E(t')$ is the adiabatic energy, an extra phase γ occurs in $\psi(t)$ which is independent of how slowly the path is traversed. $\gamma(t)$ satisfies

$$d\gamma(t)/dt = i\langle \psi(t)|d\psi(t)/dt\rangle \;. \tag{5}$$

From Eq. (2),

$$\frac{d\psi_m^{+z_0}}{dt} = N_+ \sum_i \frac{d}{dt}\ln[z_i - z_0(t)]\psi_m^{+z_0}, \tag{6}$$

so that

$$\frac{d\gamma}{dt} = iN_+^2 \left\langle \psi_m^{+z_0} \middle| \frac{d}{dt}\sum_i \ln(z_i - z_0) \middle| \psi_m^{+z_0}\right\rangle. \tag{7}$$

Since the one-electron density in the presence of the quasihole is given by

$$\rho^{+z_0}(z) = \langle \psi_m^{+z_0} | \sum_i \delta(z_i - z) | \psi_m^{+z_0} \rangle, \tag{8}$$

we have

$$\frac{d\gamma}{dt} = i \int dx\, dy\, \rho^{+z_0}(z) \frac{d}{dt} \ln[z - z_0(t)], \tag{9}$$

where $z = x + iy$. We write $\rho^{+z_0}(z) = \rho_0 + \delta\rho^{+z_0}(z)$, with $\rho_0 = \nu\phi/\phi_0$. Concerning the ρ_0 term, if z_0 is integrated in a clockwise sense around a circle of radius R, values of $|z| < R$ contribute $2\pi i$ to the integral while $|z| > R$ contributes zero. Therefore, this contribution to γ is given by

$$\gamma_0 = i \int_{|r| < R} dx\, dy\, \rho_0 2\pi i = -2\pi \langle n \rangle_R = -2\pi\nu\phi/\phi_0, \tag{10}$$

where $\langle n \rangle_R$ is the mean number of electrons in a circle of radius R. Corrections from $\delta\rho$ vanish as $(a_0/R)^2$, where $a_0 = (\hbar c/eB)^{1/2}$ is the magnetic length. This term corresponds to the finite size of the hole.

Comparing with Eq. (4), we find $e^* = \nu e$, in agreement with Laughlin's result. A similar analysis shows that the charge of the quasiparticle $\psi_m^{-z_0}$ is $-e^*$.

To determine the statistics of the quasiparticles, we consider the state with quasiholes at z_a and z_b,

$$\psi_m^{z_a, z_b} = N_{ab} \prod_i (z_i - z_a)(z_i - z_b) \psi_m. \tag{11}$$

As above, we adiabatically carry z_a aroound a closed loop of radius R. If z_b is outside the circle $|z_b| = R$ by a distance $d >> a_0$, the above analysis for γ is unchanged, i.e., $\gamma = -2\pi\nu\phi/\phi_0$. If z_b is inside the loop with $|z_b| - R << -a_0$, the change of $\langle n \rangle_R$ is $-\nu$ and one finds the extra phase $\Delta\gamma = 2\pi\nu$. Therefore, when a quasiparticle adiabatically encircles another quasiparticle an extra "statistical phase"

$$\Delta\gamma = 2\pi\nu \tag{12}$$

is accumulated.[10] For the case $\nu = 1$, $\Delta\gamma = 2\pi$, and the phase for interchanging quasiparticles is $\Delta\gamma/2 = \pi$ corresponding to Fermi statistics. For ν noninteger, $\Delta\gamma$ corresponds to fractional statistics, in agreement with the conclusion of Halperin.[11] Clearly, when ν is noninteger the change of phase $\Delta\gamma$ when a third quasiparticle is in the vicinity will depend on the adiabatic path taken by the quasiparticles as they are interchanged and the pair permutation definition used for Fermi and Bose statistics no longer suffices.

A convenient method for including the statistical phase $\Delta\gamma$ is by adding to the actual vector potential $\vec{A}_0$ a "statistical" vector potential $\vec{A}_\phi$ which has no independent dynamics. $\vec{A}_\phi$ is chosen such that

$$(e^*/\hbar c) \oint \vec{A}_\phi \cdot d\vec{l} = \Delta\gamma = 2\pi\nu, \tag{13}$$

when z_a encirlces z_b. One finds this fictious $\vec{A}_\phi$ to be

$$\vec{A}_\phi(\vec{r} - \vec{r}_b) = \frac{\phi_0 \hat{z} \times (\vec{r} - \vec{r}_b)}{2\pi |\vec{r} - \vec{r}_b|^2} \tag{14}$$

if the quasiparticles are treated as bosons and $\phi_0 \rightarrow \phi_0(1 - 1/\nu)$ if they are treated as fermions. Thus, the peculiar statistics can be replaced by a more complicated effective Lagrangian describing particles with conventional statistics.[12]

Finally, we note that if one pierces the plane with a physical flux tube of magnitude ϕ, the above arguments suggest that a charge $\nu e\phi/\phi_0$ is accumulated around the tube, regardless of whether ϕ/ϕ_0 is equal to the ratio of integers.

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