Claus Kiefer

Statement

and

Readings

DECOHERENCE IN COSMOLOGY

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Assuming the universal validity of quantum theory, the quantum-to-classical transition is also of crucial importance in cosmology. Firstly, any linear theory of quantum gravity predicts superpositions of different metrics even at the macroscopic level. Secondly, primordial fluctuations in the early Universe, out of which galaxies and clusters of galaxies are expected to develop, are of a genuine quantum nature. In my talk, I shall discuss both cases and show how and to which extent classical behaviour emerges through decoherence. The emphasis is on the main conceptual aspects rather than on technical issues.

4 Decoherence in Quantum Field Theory and Quantum Gravity

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4.2 Decoherence and the gravitational field[1]

4.2.1 Emergence of classical spacetime

According to the Copenhagen interpretation of quantum theory, the existence of a classical world is needed from the outset in order to interpret quantum theory. Appropriate classical apparata are assumed to *define* the occurrence of quantum phenomena. The presence of such classical measurement agencies seems to be possible only if spacetime exists as a classical entity.

The discussion of the previous chapters has, however, convincingly demonstrated that quantum theory has a much wider range of applicability than the pioneers had imagined. Classical properties are not intrinsic to objects but emerge through the irreversible interaction with the environment. The experiments discussed in Chap. 3 are an impressive confirmation of this idea.

What about the structure of spacetime itself? Before the advent of the general theory of relativity, spacetime was considered to be a given, non-dynamical background structure. This is also the case in quantum field theories such as QED (Sect. 4.1). In general relativity, however, the geometry of spacetime is associated with the gravitational field and thereby becomes *dynamical*. If the gravitational field is fundamentally described by quantum theory, then spacetime cannot be a classical entity.

But has gravity to be described by quantum theory? Quite generally, it does not seem possible to find a fundamental hybrid description that couples a quantum system to a classical system in a consistent way (Kiefer 2003). This does of course not mean that there exists no *effective* theory which couples quantum to classical systems. For example, one can develop a formalism in which a decohered ("classical") system is coupled to a quantum system that does not exhibit decoherence (Halliwell 1998).

As has already been mentioned, it was important already during the early discussions between Einstein and Bohr to apply the uncertainty relations to macroscopic objects (screens, photographic plates etc.) in order to save them

[1] Extract from Chapter 4 of *Decoherence and the Appearance of a Classical World in Quantum Theory*, by E. Joos, H. D. Zeh, C. Kiefer, D. Giulini, J. Kupsch, and I.-O. Stamatescu (Springer, Berlin, 2003).

for microscopic systems. This is reasonable because macroscopic objects are composed of atoms. Such *consistency arguments* are at the heart of these discussions. At the Solvay conference in 1930, Bohr and Einstein had a debate concerning the time-energy uncertainty relation, $\Delta E \Delta t \geq \hbar/2$. In the discussion, Bohr had to invoke general relativity to counter Einstein's objections. But only very little structure from general relativity does in fact enter the argument; it is only the equivalence principle and therefore the curved nature of spacetime, from which the redshift of light follows as a consequence. The redshift may be derived by just applying the energy law to the expression $\hbar\omega$ for the energy of a photon. One could thus phrase Bohr's argument in the way that a violation of the uncertainty relation would entail a violation of energy conservation.

In fact, the possible violation of conservation laws often plays an important role in such consistency arguments. Eppley and Hannah (1977), for example, consider the interaction of classical gravitational waves with quantum systems. They find, as a consequence, a violation of either momentum conservation or the uncertainty relations for the quantum system, or the occurrence of signals faster than light. Since not many peculiarities of the gravitational field enter their discussion, these results hold also for other systems such as the electromagnetic field. This type of arguments is certainly enforced for the gravitational field due to its coupling to *all* other degrees of freedom. Taking then the quantum nature of the gravitational field for granted, one would expect that efficient decoherence results from this universal coupling for both the gravitational field and other variables.

In a heuristic example, where quantum theory is applied to Newtonian gravity, one finds that the gravitational field is decohered by its action with quantum matter (Joos 1986b). Suppose that a (homogeneous) gravitational field within a box of side length L is in a quantum superposition of different strengths, i.e.

$$|\psi\rangle = c_1|g\rangle + c_2|g'\rangle, \ g \neq g'. \tag{4.1}$$

A particle with mass m in a state $|\chi\rangle$, which moves through this volume, "measures" the value of g, since its trajectory depends on the metric, yielding the total state

$$|g\rangle|\chi_g(t)\rangle \ . \tag{4.2}$$

This correlation destroys the coherence between g and g', and the reduced density matrix can be estimated to assume the following form after many such interactions are taken into account:

$$\rho(g, g', t) = \rho(g, g', 0) \exp\left(-\Gamma(g-g')^2 t\right) \ , \tag{4.3}$$

where

$$\Gamma = nL^4 \left(\frac{\pi m}{2k_B T}\right)^{3/2} \ ,$$

for a gas with particle density n and temperature T. For example, air under ordinary conditions, and $L = 1$ cm, $t = 1$ s yields a remaining coherence width of $\Delta g/g \approx 10^{-6}$.

One can give quite general arguments that the gravitational field is fundamentally of quantum nature (Kiefer 2000, 2003):

- *Singularity theorems of general relativity*: Under very general conditions, the occurrence of a singularity, and therefore the breakdown of the unquantised theory, is unavoidable. A more fundamental theory is therefore needed to overcome this breakdown, and the natural expectation is that this fundamental theory is a quantum theory of gravity. This is similar to ordinary quantum theory preventing the singularity that classical electromagnetism would predict for atoms.
- *Initial conditions in cosmology*: This is related to the singularity theorems which predict the existence of a "big bang" where the known laws of physics break down. To fully understand the evolution of our Universe, its initial state must be amenable to a physical description.
- *Unification*: Apart from general relativity, all known fundamental theories are *quantum* theories. It would thus seem awkward if gravity, which couples to all other fields, should remain the only classical entity in a fundamental description.
- *Gravity as a regulator*: Many models indicate that the consistent inclusion of gravity in a quantum framework would automatically eliminate the divergences that plague ordinary quantum field theory.
- *Problem of time*: In ordinary quantum theory, the presence of an external time parameter t is crucial for the interpretation of the theory: "Measurements" take place at a certain time, matrix elements are evaluated at fixed times, and the norm of the wave function is conserved *in* time. Since in general relativity, on the other hand, time as part of spacetime is a dynamical quantity (as defined by the metric), both concepts of time must be modified at a fundamental level.

But what does the "quantisation" of spacetime mean? In other words, to which classical structures does one have to apply the superposition principle, while the rest remains classical? Isham (1994) presents the following hierarchy of structures where this decision can be made at each level:

$$\boxed{\text{Lorentzian manifold}}$$
$$\downarrow$$
$$\boxed{\text{Causal manifold (``light cone'')}}$$
$$\downarrow$$
$$\boxed{\text{Differentiable manifold}}$$
$$\downarrow$$
$$\boxed{\text{Topological space}}$$
$$\downarrow$$
$$\boxed{\text{Set of events}}$$

A straightforward quantisation of general relativity, for example, would dissolve spacetime as a fundamental classical entity, but would retain a fixed three-dimensional manifold in the formalism. This *canonical approach* is briefly described in the next subsection and will be the basis for the calculations presented below. Path integration, for example, would entail a superposition of different manifolds. This should also be true in a "theory of everything" (for which superstring theory is a candidate) which encompasses all interactions of Nature in a single quantum framework. In such a fundamental theory it is probably only very little structure, if any, that remains classical, although this is not yet clear, cf. Seiberg and Witten (1999).

Quantum effects of gravity are expected to become relevant at the Planck scale. This is the scale where, for an elementary particle, Schwarzschild radius and Compton wavelength coincide. The Planck mass is given by

$$m_P = \sqrt{\frac{\hbar c}{G}} \approx 10^{-5} \text{ g} , \tag{4.4}$$

while Planck length and Planck time are given by the following expressions, respectively,

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-33} \text{ cm} , \quad t_P = \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-44} \text{ s} . \tag{4.5}$$

As we discuss at length in this volume, quantum effects are not a priori restricted to a particular scale. In Chap. 3 we have demonstrated that it is not the large mass by itself that provokes classical behaviour for a quantum object, but its interaction (whose strength of course depends on the mass) with the environment. Analogously, it is not the smallness of the Planck length by itself that a priori prevents quantum-gravity effects to occur at larger scales. The classical *appearance* of spacetime at larger scales should again be due to the unvoidable interaction with other degrees of freedom. It is for this reason that we can restrict ourselves in the following discussion to canonical quantum gravity, since this should be valid as an *effective* theory for scales $l \gg l_P$, independent of whether this theory is also valid at the Planck scale itself or not (in the latter case a unified theory such as string theory must be invoked).

We mention that gravity is assigned a fundamental role also in approaches which *modify* the formalism of quantum theory, see e.g. Károlyházy *et al.* (1986), Penrose (1986), as well as Chap. 8, but this will not be considered in this chapter.

4.2.2 The formalism of quantum cosmology

The basic intention in the canonical approach to quantum gravity is to derive equations for wave functionals on an appropriate configuration space, analogously to the Schrödinger picture in quantum mechanics. Technically, this is achieved by foliating, in the classical theory, the classical spacetime into spatial hypersurfaces and choosing the *spatial* metric as a canonical variable (the "q"). In the spacetime which is classically constructed by dynamically developing the initial data on a particular hypersurface, the canonical momentum is linearly related to the embedding of the hypersurfaces into spacetime. (In the case of a Friedmann universe, the radius, a, is the configuration variable, while the canonical momentum corresponds to the Hubble parameter.) The postulate of nontrivial commutation relations between these quantities in quantum gravity then means that spacetime is no longer a fundamental concept, since one cannot specify both the spatial metric and the embedding. The role of spacetime is taken over by the space of all three-dimensional geometries, which is called *superspace* and which serves as the configuration space for the theory. For a detailed physical introduction into these concepts we refer to Zeh (2001); the details of the canonical formalism are presented, for example, by Wald (1984). The central kinematical quantity is thus a wave functional defined on superspace and on matter field degrees of freedom. It is often labeled $\Psi[{}^3\mathcal{G}, \Phi]$, where ${}^3\mathcal{G}$ stands for "three-dimensional geometry" (to express the fact that this wave functional is independent of particular coordinates on the three-dimensional space, as being guaranteed by the three "momentum constraints" of general relativity), and Φ symbolically denotes all non-gravitational fields. The invariance of general relativity (called invariance under coordinate transformations or under diffeomorphisms) leads to the presence of constraints: the total Hamiltonian must vanish.[2] In the quantum theory, the constraints are implemented à la Dirac as restrictions on physically allowed wave functionals. The wave functional then obeys the Wheeler-DeWitt equation (DeWitt 1967, Wheeler 1968),

$$H\Psi = 0, \tag{4.6}$$

where H denotes the full Hamiltonian for gravity and other fields. In classical general relativity, spacetimes can be parametrised by some arbitrary time coordinates (which have lost their absolute status). Since due to the uncertainty relations no spacetimes exist anymore on the level of quantum gravity (only a wave function for spatial metrics), there is no time parameter available to parametrise them – the Wheeler-DeWitt equation is "timeless". This gives rise to the *problem of time* in quantum gravity which is extensively discussed in the literature, see e.g. Barbour (1994a,b), Isham (1992), Kuchař (1992), Zeh (1986, 2001), Kiefer and Zeh (1995), and Kiefer (2000, 2003).

We have to emphasise that this approach at present exists only on a formal level, since the explicit treatment of (4.6) is unclear.[3] In this respect the discussion in the present section is different from the rest of the book and

[2] We consider only the case of spatially closed hypersurfaces. In the asymptotically flat case, the total Hamiltonian can be written as a surface integral.

[3] It is known that (4.6) does not give rise to a unitary evolution in a Fock space built over three-dimensional slices.

should be considered of heuristic value only. However, from general arguments like reparametrisation invariance one would expect the fundamental equation to be of the constraint form (4.6), although the exact form of H may be different. Therefore, the main *interpretational* part of the discussion in this section would remain unaffected, and only the details of the calculations would have to be changed.

The main features of the canonical approach can already be recognised in a simple two-dimensional model – a closed Friedmann universe characterised by its scale factor a, containing a homogeneous massive scalar field φ as a representation of matter, cf. Kiefer (1988) and Halliwell (1991). Taking the units $2G = 3\pi$, the classical action for this model is the sum of the gravitational part and the matter part,

$$\begin{aligned} S &= \int dt\ L(a, \dot{a}, \varphi, \dot{\varphi}, N) \\ &\equiv \frac{1}{2}\int dt\ Na^3\left(-\frac{\dot{a}^2}{N^2a^2} + \frac{\dot{\varphi}^2}{N^2} + \frac{1}{a^2} - m^2\varphi^2\right) . \end{aligned} \tag{4.7}$$

This action is invariant with respect to arbitrary reparametrisations of the time variable t, a fact which is encoded in the presence of the non-dynamical *lapse function* N which appears undifferentiated in the action. A characteristic feature of the gravitational field is the occurrence of an indefinite kinetic term in the action.

The standard canonical formalism proceeds with the definition of the canonical momenta,

$$p_N = \frac{\partial L}{\partial \dot{N}} = 0,\ p_a = \frac{\partial L}{\partial \dot{a}} = -\frac{a\dot{a}}{N},\ p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \frac{a^3\dot{\varphi}}{N} . \tag{4.8}$$

The canonical Hamiltonian is then given by

$$\begin{aligned} H &= p_N\dot{N} + p_a\dot{a} + p_\varphi\dot{\varphi} - L \\ &= \frac{N}{2}\left(-\frac{p_a^2}{a} + \frac{p_\varphi^2}{a^3} - a + m^2\varphi^2a^3\right) \\ &\equiv \frac{N}{2}G^{AB}p_Ap_B + V(a,\varphi) . \end{aligned} \tag{4.9}$$

The important point is that $p_N = 0$ is a constraint that should hold at all times. Therefore, from Hamiton's equations of motion one gets $\partial H/\partial N = 0$ which gives the constraint

$$H = 0 \Leftrightarrow \dot{a}^2 = -1 + a^2(\dot{\varphi}^2 + m^2\varphi^2) . \tag{4.10}$$

This is nothing but the classical Friedmann equation which is well known from cosmology. Variation of (4.7) with respect to a and φ give the classical equations of motion. The equation for φ, in particular, reads

$$\ddot{\varphi} + \frac{3\dot{a}}{a}\dot{\varphi} + m^2\varphi = 0 . \tag{4.11}$$

This is the Klein-Gordon equation for a homogeneous field in an evolving universe, whose effect on φ is the second ("friction") term.

Following Dirac's procedure, the classical constraint (4.10) is then turned into the Wheeler-DeWitt equation (4.6). Using a particular factor ordering,[4] the explicit form of this equation in the present model reads

$$H\psi \equiv \left(\hbar^2 a \frac{\partial}{\partial a}\left(a\frac{\partial}{\partial a}\right) - \hbar^2 \frac{\partial^2}{\partial\varphi^2} + m^2\varphi^2 a^6 - a^4\right)\psi(a,\varphi) = 0 \ . \qquad (4.12)$$

Note that the indefiniteness of the kinetic term has led to a *hyperbolic* equation for ψ – in contrast to the Schrödinger equation. In the next subsection, a more complicated model is used in which the variables a and φ play the role of the background, supplemented by additional degrees of freedom ("higher multipoles") $\{f_n\}$. (In the following we set again $\hbar = 1$.)

The Wheeler-DeWitt equation (4.6), (4.12) does not contain a classical time parameter. This is not surprising, since the classical metric is known to determine time. An *approximate* concept of time-dependence of a wave function can be recovered in a Born-Oppenheimer type of approximation scheme in which part of the degrees of freedom are semiclassical (given by WKB wave functions), while the rest is fully quantum. This limit is obtained, for example, if the full wave functional in (4.6) is of the form

$$\Psi[{}^3\mathcal{G},\Phi] \approx \sum_{(n)} C_{(n)}[{}^3\mathcal{G}]e^{iS_0^{(n)}[{}^3\mathcal{G}]/G\hbar}\psi^{(n)}[{}^3\mathcal{G},\Phi] \equiv \sum_{(n)} \psi_0^{(n)}\psi^{(n)}, \qquad (4.13)$$

where the $S_0^{(n)}$ are solutions to the gravitational Hamilton–Jacobi equations which are fully equivalent to Einstein's field equations. The gravitational part of the total state is thus treated semiclassically. The semiclassical part may also comprise part of the matter degrees of freedom. In fact, in the discussion of decoherence in Sect. 4.2.3, the scalar field φ will belong to this part. Note the analogy to Equation (**??**) discussed in the last section.

The sum in (4.13) runs over a whole set of indices (n) (which may also be continuous). It turns out that the matter states $\psi^{(n)}$ obey the following approximate equation in each component,

$$i\nabla S_0^{(n)} \cdot \nabla\psi^{(n)} \approx H_m^{(n)}\psi^{(n)}, \qquad (4.14)$$

where $H_m^{(n)}$ denotes the Hamiltonian for the non-gravitational fields (which of course depends on the particular solution $S_0^{(n)}$ chosen for the gravitational field). Note the analogy of $H_m^{(n)}$ to H_ϕ discussed in the last section. The expression $\nabla S_0^{(n)} \cdot \nabla \equiv \partial/\partial t^{(n)}$ is a directional derivative in the gravitational part of the full configuration space, which parametrises the family of

[4] The chosen factor ordering is given by the Laplace-Beltrami operator in the configuration space spanned by a and φ.

classical spacetimes described by $S_0^{(n)}$. The parameters $t^{(n)}$ are often called *WKB times* – they control the "dynamical evolution" of the states $\chi^{(n)}$ along the WKB trajectories. Equation (4.14) is thus nothing but the Schrödinger equation, while t represents our phenomenological time. Details of the semiclassical approximation to quantum gravity are described in Kiefer (1994), see also Kiefer (1993), Giulini and Kiefer (1995).

We note that due to the central input of the Born-Oppenheimer expansion the situation here is analogous to that of Sect. 4.1.2 only ("measurement" of fields by charges), since the reverse effect (which would here correspond to "measurement" of matter by the gravitational field) is too weak to become important.

4.2.3 Decoherence in quantum cosmology

In quantum cosmology, all variables are fundamentally quantum and there is no classical spacetime. How does a classical spacetime emerge? It has been suggested that global degrees of freedom such as the volume of the Universe appear classical after the interaction with other degrees of freedom is taken into account (Zeh 1986). The role of such additional variables may be played by density fluctuations and gravitational waves. All these degrees of freedom are of course within the Universe, but they are "environmental" to the volume-degree of freedom in configuration space. From the viewpoint of a "local" observer who can measure the size of the Universe but has no access to small fluctuations, these other degrees of freedom have to be traced over. In this sense they are able to produce decoherence for the volume degree of freedom. We have emphasised before that the issue of classicality only arises *after* a quantum system has been chosen, for which the straightforward application of the superposition principle would lead to a macroscopically entangled state. In a sense, a classical spacetime thus arises by a "self-measurement" of the Universe.

Calculations for decoherence in quantum cosmology can be done with the help of the Wheeler-DeWitt equation (4.6), see Kiefer (1987). As a necessary prerequisite, the semiclassical approximation to quantum gravity is employed, in which an approximate Schrödinger equation is recovered for the cosmological fluctuations (see Sect. 4.2.2). The time parameter corresponding to this equation is defined by the semiclassical degrees of freedom (Halliwell and Hawking 1985). In Kiefer (1987) the relevant system was taken to be the scale factor ("radius") a of the Universe together with a homogeneous scalar field φ, cf. the model discussed in Sect. 4.2.2. The field φ plays a crucial role in modern cosmological theories where an exponential, "inflationary", expansion is assumed to have happened in an early phase of the Universe, starting about 10^{-33} s after the big bang. It is in fact the "inflaton field" φ itself that causes inflation. The inhomogeneous modes of the gravitational field and the scalar field (gravitational waves and density fluctuations) can then be shown to *decohere* the global variables a and φ.

An open problem in these early papers was the issue of regularisation; the number of fluctuations is infinite and would cause divergences, which is why a cutoff was suggested. The issue was again addressed in Barvinsky *et al.* (1999a) where a physically motivated regularisation scheme was introduced. In the following we shall briefly review this approach.

As a (semi)classical solution for a and φ one may use

$$\varphi(t) \approx \varphi, \tag{4.15}$$

$$a(t) \approx \frac{1}{H(\varphi)} \cosh H(\varphi)t \ , \tag{4.16}$$

where $H(\varphi) = 8\pi V(\varphi)/3m_P^2$ is the Hubble parameter generated by the inflaton potential $V(\varphi)$. It is approximately constant during the inflationary phase in which φ slowly "rolls down" the potential. We take into account fluctuations of a field $f(t, \mathbf{x})$ which can be a field of any spin. Space is assumed to be a closed three-sphere, so $f(t, \mathbf{x})$ can be expanded into a discrete series of spatial orthonormal harmonics $Q_n(\mathbf{x})$,

$$f(t, \mathbf{x}) = \sum_n f_n(t) Q_n(\mathbf{x}) \ . \tag{4.17}$$

One can thus represent the fluctuations by the degrees of freedom f_n.

Our intention now is to solve the Wheeler-DeWitt equation (4.6) in the semiclassical approximation. This leads to the following solution:

$$\Psi(t|\varphi, f) = \frac{1}{\sqrt{v_\varphi^*(t)}} e^{-I(\varphi)/2 + iS_{\rm cl}(t,\varphi)} \prod_n \psi_n(t, \varphi|f_n) \ . \tag{4.18}$$

The time t that appears here is the semiclassical ("WKB") time and is defined by the background-degrees of freedom a and φ through the "eikonal" $S_{\rm cl}$ which is a solution of the Hamilton-Jacobi equation; t is formally identical with the time that appears in the classical equations (4.15) and (4.16). Since φ is thus determined by a, only one variable (a *or* φ) occurs in the argument of Ψ. The wave functions ψ_n for the fluctuations f_n obey each an approximate Schrödinger equation (4.14) with respect to t, and their Hamiltonian H_n has the form of a ("time-dependent") harmonic-oscillator Hamiltonian. The first exponent contains the euclidean action $I(\varphi)$ from the classically forbidden region (the "De Sitter instanton") and is independent of t. Its form depends on the *boundary conditions* imposed. In the present case the so-called Hartle-Hawking condition is chosen, see e.g. Halliwell (1991), which amounts to $I(\varphi) \approx -3m_P^4/8V(\varphi)$. The detailed form is, however, not necessary for the discussion below. The function $v_\varphi(t)$ is the so-called basis function for φ and is a solution of the classical equation of motion. In the following we shall choose units such that $G = c = \hbar = 1$.

For the ψ_n we shall take – in analogy to (**??**) – Gaussian states that correspond to the so-called De Sitter-invariant vacuum state (Starobinsky 1979, Allen 1985). This is the maximally symmetric state which possesses properties very similar to the standard vacuum state in Minkowski space. (In the massless case, this state is invariant only under a subgroup of the De Sitter group.) It is given by

$$\psi_n(t,\varphi|f_n) = \frac{1}{\sqrt{v_n^*(t)}} \exp\left(-\frac{1}{2}\Omega_n(t)f_n^2\right), \tag{4.19}$$

$$\Omega_n(t) = -ia^3(t)\frac{\dot{v}_n^*(t)}{v_n^*(t)} . \tag{4.20}$$

The functions v_n are the basis functions of the De Sitter-invariant vacuum state; they satisfy the classical equation of motion

$$F_n\left(\frac{d}{dt}\right)v_n \equiv \left(\frac{d}{dt}a^3\frac{d}{dt} + a^3m^2 + a(n^2-1)\right)v_n = 0 \tag{4.21}$$

with the boundary condition that they should correspond to a standard Minkowski positive-frequency function for constant a. In the simple special case of a spatially flat section of De Sitter space one would have

$$av_n = \frac{e^{-in\eta}}{\sqrt{2n}}\left(1 - \frac{i}{n\eta}\right) , \tag{4.22}$$

where η is the *conformal time* defined by $ad\eta = dt$. We note that the corresponding *negative*-frequency function enters the exponent of the Gaussian, see (4.20).

An important property of these vacuum states is that their norm is conserved *along any semiclassical solution* (4.15), (4.16),

$$\left\langle\psi_n|\psi_n\right\rangle \equiv \int df_n|\psi_n(f_n)|^2 = \sqrt{2\pi}[\Delta_n(\varphi)]^{-1/2}, \tag{4.23}$$

$$\Delta_n(\varphi) \equiv ia^3(v_n^*\dot{v}_n - \dot{v}_n^*v_n) = \text{constant} . \tag{4.24}$$

Note that $\Delta_n(\varphi)$ is just the (constant) Wronskian corresponding to (4.21). (The corresponding Wronskian for the homogeneous mode φ is $\Delta_\varphi \equiv ia^3(v_\varphi^*\dot{v}_\varphi - \dot{v}_\varphi^*v_\varphi)$.) We must emphasise that Δ_n is a nontrivial function of the background variable φ, since it is defined on full configuration space and not only along semiclassical trajectories (it gives the weights in the "Everett branches".) It is therefore *not* possible to normalise the ψ_n artificially to one, since this would be inconsistent with respect to the full Wheeler-DeWheeler equation (Barvinsky *et al.* 1999a).

The solution (4.18) forms the basis for our discussion of decoherence. Since the $\{f_n\}$ are interpreted as the environmental degrees of freedom, they have to be integrated out to get the reduced density matrix for φ or a (a and φ can be used interchangeably, since they are connected by t). The reduced density matrix thus reads

$$\rho(t|\varphi,\varphi') = \int df\Psi(t|\varphi,f)\Psi^*(t|\varphi',f) , \tag{4.25}$$

where Ψ is given by (4.18), and it is understood that $df = \prod_n df_n$. After the integration one finds

$$\rho(t|\varphi,\varphi') = C\frac{1}{\sqrt{v_\varphi^*(t)v_\varphi'(t)}}\exp\Big[-\frac{1}{2}I - \frac{1}{2}I' + i(S-S')\Big] \times \prod_n \Big[v_n^* v_n'(\Omega_n + \Omega_n'^*)\Big]^{-1/2}, \tag{4.26}$$

where C is a numerical constant. The diagonal elements $\rho(t|\varphi,\varphi)$ describe the probabilities for certain values of the inflaton field to occur. In an appropriate model, one can find that these probabilities are peaked at the onset of inflation around values of φ that lead to phenomenologically satisfying results (for example, with respect to structure formation) without having to invoke the anthropic principle, see Barvinsky *et al.* (1999b) and the references therein.

It is convenient to rewrite the expression for the density matrix (4.26) in the form

$$\rho(t|\varphi,\varphi') = C\frac{\Delta_\varphi^{1/4}\Delta_\varphi'^{1/4}}{\sqrt{v_\varphi^*(t)v_\varphi'(t)}}\exp\left(-\frac{1}{2}\boldsymbol{\Gamma} - \frac{1}{2}\boldsymbol{\Gamma}' + i(S-S')\right) \times \boldsymbol{D}(t|\varphi,\varphi'), \tag{4.27}$$

where

$$\boldsymbol{\Gamma} = I(\varphi) + \boldsymbol{\Gamma}_{1-\mathrm{loop}}(\varphi) \tag{4.28}$$

is the full Euclidean effective action including the classical part and the one-loop part. The latter comes from the next-order WKB approximation and is important for the normalisability of the wave function with respect to φ. The last factor in (4.27) is the *decoherence factor*

$$\boldsymbol{D}(t|\varphi,\varphi') = \prod_n \left(\frac{4\mathrm{Re}\,\Omega_n\,\mathrm{Re}\,\Omega_n'^*}{(\Omega_n + \Omega_n'^*)^2}\right)^{1/4}\left(\frac{v_n}{v_n^*}\frac{v_n'^*}{v_n'}\right)^{1/4}. \tag{4.29}$$

It is equal to one for coinciding arguments. While the decoherence factor is time-dependent, the one-loop contribution to (4.27) does not depend on time and may play only a role at the onset of inflation. In a particular model with non-minimal coupling (Barvinsky *et al.* 1997), the size of the non-diagonal elements is at the onset of inflation approximately equal to those of the diagonal elements. The Universe would thus be essentially quantum at this stage, i.e. in a non-classical state.

The amplitude of the decoherence factor can be rewritten in the form

$$|\boldsymbol{D}(t|\varphi,\varphi')| = \exp\frac{1}{4}\sum_n \ln\frac{4\mathrm{Re}\,\Omega_n\,\mathrm{Re}\,\Omega_n'^*}{|\Omega_n + \Omega_n'^*|^2}. \tag{4.30}$$

The convergence of this series is far from being guaranteed. Moreover, the divergences might not be renormalisable by local counterterms in the bare quantised action. We shall now analyse this question in more detail.

We start with a minimally coupled massive scalar field. Equation (4.21) for the basis functions reads

$$\frac{d}{dt}\left(a^3\frac{dv_n}{dt}\right)+a^3\left(\frac{n^2-1}{a^2}+m^2\right)v_n=0\ . \tag{4.31}$$

The appropriate solution to this equation is (Barvinsky *et al.* 1992)

$$v_n(t)=(\cosh Ht)^{-1}P^{-n}_{-\frac{1}{2}+i\sqrt{m^2/H^2-9/4}}(i\sinh Ht)\ , \tag{4.32}$$

where P denotes an associated Legendre function of the first kind. The expansion of (4.32) for large masses was derived in Barvinsky *et al.* (1992). The corresponding expression for (4.20) is given by

$$\Omega_n=a^2\left[\sqrt{n^2+m^2a^2}+i\sinh Ht\left(1+\frac{1}{2}\frac{m^2a^2}{n^2+m^2a^2}\right)\right]+O\left(\frac{1}{m}\right) \tag{4.33}$$

The leading contribution to the amplitude of the decoherence factor is therefore

$$\ln|\boldsymbol{D}(t|\varphi,\varphi')|\simeq\frac{1}{4}\sum_{n=0}^{\infty}n^2\ln\frac{4a^2a'^2\sqrt{n^2+m^2a^2}\,\sqrt{n^2+m^2a'^2}}{\left(a^2\sqrt{n^2+m^2a^2}+a'^2\sqrt{n^2+m^2a'^2}\right)^2} \tag{4.34}$$

The first term, n^2, in the sum comes from the degeneracy of the eigenfunctions. This expression has divergences which *cannot* be represented as additive functions of a and a'. This means that no one-argument counterterm to $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}'$ in (4.27) can cancel these divergences of the amplitude (Paz and Sinha 1992). One might try to apply standard regularisation schemes from quantum field theory, such as dimensional regularisation. The corresponding calculations have been performed in Barvinsky *et al.* (1999a) and will not be given here. The important result is that, although they render the sum (4.34) convergent, they lead to a *positive* value of this expression. This means that the decoherence factor would diverge for $(\varphi-\varphi')\to\infty$ and thus spoil one of the crucial properties of a density matrix – the boundedness of $\mathrm{tr}\,\hat{\rho}^2$. The dominant term in the decoherence factor would read

$$\ln|\boldsymbol{D}|=\frac{\pi}{24}(ma)^3+O(m^2),\ \ a\gg a' \tag{4.35}$$

and would thus be unacceptable for a density matrix. Reduced density matrices are usually not considered in quantum field theory, so this problem has not been encountered before. A behaviour such as in (4.35) is even obtained in the case of massless conformally invariant fields, for which one would expect a decoherence factor equal to one, since they decouple from the gravitational background. How, then, does one have to proceed in order to obtain a sensible regularisation?

The crucial point is to perform a *redefinition* of environmental fields and to invoke a physical principle to fix this redefinition. The situation is somewhat analogous to the treatment of the S-matrix in quantum field theory: off-shell S-matrix and effective action depend on the parametrisation of the quantum fields (Vilkovisky 1984), in analogy to the non-diagonal elements of the reduced density matrix. In Laflamme and Louko (1991) and Kiefer (1992) it has been proposed within special models to rescale the environmental fields by a power of the scale factor. It was therefore suggested in Barvinsky *et al.* (1999a) to redefine the environmental fields by a power of the scale factor that corresponds to the conformal weight of the field (which is defined by the invariance of the conformally invariant wave equation). For a scalar field in four spacetime dimensions this amounts to a multiplication by a:

$$v_n(t) \to \tilde{v}_n(t) = a\, v_n(t) \;, \tag{4.36}$$

$$\tilde{\Omega}_n \quad = -ia\frac{d}{dt}\ln \tilde{v}_n^* \;. \tag{4.37}$$

An immediate test of this proposal is to see whether the decoherence factor is equal to one for a massless conformally invariant field. In this case, the basis functions and frequency functions read, respectively,

$$\tilde{v}_n^*(t) = \left(\frac{1+i\sinh Ht}{1-i\sinh Ht}\right)^{\frac{n}{2}} \;, \tag{4.38}$$

$$\tilde{\Omega}_n = -ia\frac{d}{dt}\ln \tilde{v}_n^*(t) = n \;. \tag{4.39}$$

Hence, $\tilde{\boldsymbol{D}}(t|\varphi,\varphi') \equiv 1$. The same holds also for the electromagnetic field (which in four spacetime dimensions is conformally invariant). It is interesting to note that the degree of decoherence caused by a certain field depends on the spacetime dimension, since its conformal properties are dimension-dependent.

For a massive minimally coupled field the new frequency function reads

$$\tilde{\Omega}_n = \left[\sqrt{n^2+m^2a^2} + i\sinh Ht\left(\frac{1}{2}\frac{m^2a^2}{n^2+m^2a^2}\right)\right] + O(1/m) \;. \tag{4.40}$$

Note that, in contrast to (4.33), there is no factor of a in front of this expression. Since (4.40) is valid in the large-mass limit, it corresponds to modes which evolve adiabatically on the gravitational background, the imaginary part in (4.40) describing particle creation.

It turns out that the imaginary part of the decoherence factor has at most logarithmic divergences and, therefore, affects only the phase of the density matrix. Moreover, these divergences decompose into an *additive* sum of one-argument functions and can thus be cancelled by adding counterterms to the classical action S (and S') in (4.27) (Paz and Sinha 1992). The real part is simply convergent and gives a finite decoherence amplitude. This result is formally similar to the result for the decoherence factor in QED (Kiefer 1992).

For $a \gg a'$ (far off-diagonal terms) one gets the expression

$$|\tilde{\boldsymbol{D}}(t|\varphi,\varphi')| \simeq \exp\left[-\frac{(ma)^3}{24}\left(\pi - \frac{8}{3}\right) + O(m^2)\right] . \tag{4.41}$$

Compared with the naively regularised (and inconsistent) expression (4.35), π has effectively been replaced by $8/3 - \pi$. In the vicinity of the diagonal, one obtains

$$\ln|\tilde{\boldsymbol{D}}(t|\varphi,\varphi')| = -\frac{m^3\pi a(a-a')^2}{64} , \tag{4.42}$$

a behaviour similar to (4.41).

An interesting case is also provided by minimally coupled massless scalar fields and by gravitons. They share the basis- and frequency functions in their respective conformal parametrisations:

$$\tilde{v}_n^*(t) = \left(\frac{1+i\sinh Ht}{1-i\sinh Ht}\right)^{\frac{n}{2}}\left(\frac{n-i\sinh Ht}{n+1}\right), \tag{4.43}$$

$$\tilde{\Omega}_n = \frac{n(n^2-1)}{n^2-1+H^2a^2} - i\frac{H^2a^2\sqrt{H^2a^2-1}}{n^2-1+H^2a^2}. \tag{4.44}$$

They differ only by the range of the quantum number n ($2 \le n$ for inhomogeneous scalar modes and $3 \le n$ for gravitons) and by the degeneracies of the n-th eigenvalue of the Laplacian,

$$\dim(n)_{\mathrm{scal}} = n^2 , \tag{4.45}$$

$$\dim(n)_{\mathrm{grav}} = 2(n^2-4). \tag{4.46}$$

For far off-diagonal elements one obtains the decoherence factor

$$|\tilde{\boldsymbol{D}}(t|\varphi,\varphi')| \sim e^{-C(Ha)^3}, \;\; a \gg a', \;\; C > 0 , \tag{4.47}$$

while in the vicinity of the diagonal one finds

$$|\tilde{\boldsymbol{D}}(t|\varphi,\varphi')| \sim \exp\left(-\frac{\pi^2}{32}(H-H')^2t^2e^{4Ht}\right) , \tag{4.48}$$

$$\sim \exp\left(-\frac{\pi^2H^4a^2}{8}(a-a')^2\right) , \;\; Ht \gg 1 . \tag{4.49}$$

These expressions exhibit a rapid disappearance of non-diagonal elements during the inflationary evolution.

It is interesting that the behaviour of fermions concerning decoherence is different from the behaviour of bosons (Barvinsky *et al.* 1999c). Since their conformal weight is $-3/2$ in four dimensions, the environmental fermionic fields are reparametrised by a factor $a^{-3/2}$. For $m = 0$ this does, as in the bosonic case, render the decoherence factor finite and, due to conformal invariance, makes it equal to one. The situation for $m \neq 0$ is, however, different. In spite of the conformal reparametrisation, the decoherence factor is divergent. Moreover, dimensional regularisation would again spoil crucial properties of the density matrix and make it inconsistent. There remains, however, a freedom of reparametrisation in the fermionic case (Barvinsky *et al.* 1999c): this is a Bogoliubov transformation that is analogous to a Foldy-Wouthuysen transformation in Minkowski space (the decoupling of spinor components in the nonrelativistic limit). Since it is explicitly n-dependent, it corresponds to a *nonlocal* field redefinition. Instead of m one has now an effective n-dependent mass $\tilde{m}$ depending on the transformation. How can one fix this field redefinition? In Barvinsky *et al.* (1999c) the principle was put forward that decoherence should be *minimal* in the absence of particle creation. This is already implemented in the massless case. In the massive case, it means that decoherence is absent for a stationary spacetime which exhibits no particle creation. This leads to a decoherence factor

$$|\tilde{\boldsymbol{D}}(t|\varphi,\varphi')| \sim \exp\left(-C'm^2H^2a^2(a-a')^2\right)\,, \quad C'>0\;. \tag{4.50}$$

While decoherence is thus absent in the absence of particle creation, for bosons it is minimal in the sense that it is absent in the conformally-coupled case, but still present in the massive case – the expressions (4.41) and (4.42) do not depend on H. Formally, this is due to the fact that in the fermionic case one has m^2 instead of m^3 in the exponent; since one would expect to find factors of a in the nominator of the exponent (as is suggested by the coupling in the action), they have to be accompanied by corresponding factors of H for dimensional reason. Comparing (4.50) with (4.41) and (4.42) (which are valid for $m \gg H$), one recognises that fermions are *less efficient* in producing decoherence. In the massless case, there influence is totally absent. The point that decoherence is linked with particle creation has been made before (Calzetta and Hu 1994, Hu and Matacz 1995). Using the influence-functional approach to decoherence, see Chap. 5, one can derive an explicit formula connecting the decoherence factor with the Bogoliubov coefficients describing particle creation (Hu and Matacz 1995).[5] Given a special initial state (a "vacuum"), this encodes the irreversible aspect of decoherence. In the massless bosonic case, (4.47) and (4.49), the effect may be interpreted as arising from a cutoff at a mode number $n \approx aH$, i.e., a cutoff of modes whose wavelength a/n is smaller than the Hubble scale H^{-1} (Halliwell 1989). As we shall see in the next subsection, these are exactly the modes that experience particle creation.

[5] The decoherence factor in the massive bosonic case, (4.41) and (4.42), comes from the adiabatic part of $\tilde{\Omega}_n$ and is not directly related to particle creation. This is not in conflict with Hu and Matacz (1995), since there the assumption is being made that the state separates between system and environment in the past, which is not the case here.

The above analysis of decoherence was based on the state (4.18). One might, however, start with a quantum state that is a superposition of many semiclassical components, i.e. many components of the form $\exp(iS_{\rm cl}^k)$, where each $S_{\rm cl}^k$ is a solution of the Hamilton-Jacobi equation for a and φ. Decoherence between different such semiclassical *branches* has also been the subject of intense investigation (Halliwell 1989, Kiefer 1992). The important point is that decoherence between different branches is usually weaker than the above discussed decoherence within one branch. Moreover, it usually follows from the presence of decoherence within one branch. In the special case of a superposition of (4.18) with its complex conjugate, one can immediately recognise that decoherence between the semiclassical components is smaller than within one component: in the expression (4.29) for the decoherence factor, the term $\Omega_n + \Omega_n'^*$ in the denominator is replaced by $\Omega_n + \Omega_n'$. Therefore, the imaginary parts of the frequency functions add up instead of partially cancelling each other and (4.29) becomes smaller. One also finds that the decoherence factor is equal to one for vanishing expansion of the semiclassical universe (Kiefer 1992).

We note that the decoherence between the $\exp(iS_{\rm cl})$ and $\exp(-iS_{\rm cl})$ components can be interpreted as a *symmetry breaking* analogously to the case of sugar molecules, see Sect. 3.2.4 and Chap. 9. There, the Hamiltonian is invariant under space reflections, but the state of the sugar molecules exhibits chirality. Here, the Hamiltonian in (4.6) is invariant under complex conjugation,[6] while the "actual states" (i.e., one decohering WKB component in the total superposition) are of the form $\exp(iS_{\rm cl})$ and are thus intrinsically complex. It is therefore not surprising that the recovery of the classical world follows only for complex states, in spite of the real nature of the Wheeler-DeWitt equation (see in this context Barbour 1993). Since this is a prerequisite for the derivation of the Schrödinger equation, one might even say that *time* (the WKB time parameter in the Schrödinger equation) arises from symmetry breaking.

The above considerations thus lead to the following picture. The Universe was essentially "quantum" at the onset of inflation. Mainly due to bosonic fields, decoherence set in and led to the emergence of many "quasi-classical branches" which are dynamically independent of each other. Strictly speaking, the very concept of time makes only sense after decoherence has occurred. In addition to the horizon problem etc., inflation also solves the "classicality problem". It remains of course unclear why inflation happened in the first place (if it really did). Looking back from our Universe (our semiclassical branch) to the past, one would notice that at the time of the onset of inflation our component would interfere with other components to form a timeless quantum-gravitational state. The Universe would thus cease to be transparent to earlier times (because there was no time). This demonstrates in an

[6] We ignore here alternative approaches which use a complex Hamiltonian from the very beginning (Kiefer 1993).

impressive way that quantum-gravitational effects are not restricted to the Planck scale.

It is interesting that a similar kind of constructive interference would occur near the turning point of a classically recollapsing universe (Kiefer and Zeh 1995). This is a direct consequence of the consistent way in which boundary conditions have to be imposed in this case. Again, this demonstrates that quantum effects are not restricted a priori to a particular scale and that it is a quantitative question referring to the dynamics when and to which extent classical properties emerge.

Our analysis has been restricted to the case where the "system" is taken to be a Friedmann universe containing a homogeneous scalar field. This is justified from phenomenological grounds, since our Universe appears isotropic and homogeneous on largest scales. Again, this may be traced back to the presumed occurrence of an inflationary phase and the validity of the cosmic no-hair conjecture. In spite of this, one can discuss decoherence in the context of anisotropic models, too (Gangui *et al.* 1991, Camacho and Camacho-Galván 1999), and find classical properties for the corresponding scale factors.

We want finally to stress the importance of decoherence for the *origin of irreversibility* in our Universe (Zeh 2001; Kiefer and Zeh 1995). Since the entropy of the present Universe (defined by its "relevant" degrees of freedom) is still extremely small compared to its maximal possible value (which would be achieved if the whole mass of the Universe were present in the form a black hole), the evolution of the Universe must have been started with a state of almost zero entropy (Penrose 1981). A possible explanation of this fact must necessarily invoke the fundamental quantum theory of gravity. It has been argued in the above references that a simple boundary condition at $a \to 0$ for the wave function of the Universe may be sufficient to explain the observed arrow of time, and may even lead to macroscopic quantum effects near the turning point of a classically recollapsing universe as well as for black holes. Such a boundary condition was proposed, for example, in Conradi and Zeh (1991). It roughly states that the wave function for small a depends only on a itself, but not on further degrees of freedom. This is consistent with the special form of the potential in the Wheeler-DeWitt equation. The wave function is thus independent, in this limit, of the "higher multipoles" introduced in this section. For increasing size of the Universe, the total state becomes entangled with these further degrees of freedom, and the decoherence for the "relevant subsystem" can be recognised after the "irrelevant" part is integrated out. The local entropy connected with the scale factor and other "relevant" variables, as calculated from the reduced density matrix in the standard way, $S = -k_{\mathrm{B}}\mathrm{tr}(\rho \ln \rho)$, thus *increases* and gives rise to the observed arrow of time in the Universe. An interesting consequence is the occurrence of recoherence in the case of a classically recollapsing universe (Kiefer and Zeh 1995).

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Why do cosmological perturbations look classical to us?

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According to the inflationary scenario of cosmology, all structure in the Universe can be traced back to primordial fluctuations during an accelerated (inflationary) phase of the very early Universe. A conceptual problem arises due to the fact that the primordial fluctuations are quantum, while the standard scenario of structure formation deals with classical fluctuations. In this essay we present a concise summary of the physics describing the quantum-to-classical transition. We first discuss the observational indistinguishability between classical and quantum correlation functions in the closed system approach (pragmatic view). We then present the open system approach with environment-induced decoherence. We finally discuss the question of the fluctuations' entropy for which, in principle, the concrete mechanism leading to decoherence possesses observational relevance.

Keywords: primordial fluctuations; inflation; decoherence; entropy

I. INTRODUCTION

It is often emphasized these days that the field of cosmology has entered a golden age. There is no doubt that the main reason for this statement is the accumulation of observations of ever increasing accuracy. In this way cosmological models aiming to describe the evolution of the Universe from the Big Bang until today are no longer purely speculative: their predictions can be tested and some models can indeed be ruled out.

With the advent of inflationary models, according to which the Universe underwent a phase of accelerated expansion at a very early stage, we now have at our disposal theoretical tools to apprehend such fundamental problems as the origin of cosmological perturbations and the eventual formation of large-scale structures like galaxies. There are many ways in which inflationary models address fundamental physical theories. As inflation is supposed to take place at very high energies in the early Universe, these models offer a unique window on energy scales of the order of 10^{15} GeV. Another intriguing aspect of these models is that inflationary perturbations originate from quantum fluctuations though we do not see this quantum nature in the Universe nowadays. It is this aspect of inflationary perturbations that we want to describe in our essay.

We could, of course, as well consider non-inflationary cosmological models in which perturbations are assumed to be classical from the beginning on. However, such models are plagued with problems of causality as distant points on the last-scattering surface, about 350.000 years after the Big Bang, were never in contact before. Hence the impressive homogeneity of the Cosmic Microwave Background (CMB) would have to be put in by hand in the absence of an inflationary stage. Inflationary models are thus much more natural – and they can be observationally tested.

The main part of our essay consists of four parts. We shall first give in Sec. II a brief review of inflationary cosmology and its mechanism for the generation of perturbations. We then discuss in Sec. III the quantum-to-classical transition in the closed system approach (we call it also the pragmatic view) which focusses on the indistinguishability of quantum expectation values and classical stochastic averages. Sec. IV presents the successful observational predictions which emerge from this scenario. Sec. V, then, is devoted to environmental decoherence. We discuss the problem of the classical variables (the pointer basis) as well as the entropy of the fluctuations and its observational significance. We end with a brief conclusion.

II. INFLATION

We give here a brief review of the way in which inflationary models give an elegant solution to many fundamental problems occuring in non-inflationary Big-Bang cosmology, see, for example,[1]. As we shall see, these models do also make characteristic predictions, by which we mean that in the absence of certain observable signatures most if not all inflationary models would be ruled out. We shall first describe the evolution of the homogeneous background for inflation and then turn to the generation of perturbations.

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A. Background expansion

The crucial point here is that inflation is a stage of accelerated expansion. In this stage, proper (physical) scales are stretched by a huge factor so that scales inside the Hubble radius during inflation will eventually end up at the end of inflation far outside the Hubble radius. Today these scales can correspond to cosmological scales, and typically scales corresponding to the Hubble radius today have exited the Hubble radius during inflation about 65 e-folds before the end of inflation. Typically, inflationary stages are quasi-de Sitter stages during which the Hubble parameter is nearly constant. As we shall see below, inflation provides a mechanism for the causal generation of perturbations.

It is a basic assumption that our Universe is on large scales homogeneous and isotropic. The metric is of the form

$$ds^2 = dt^2 - a^2(t)\ \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\ d\phi^2)\right]\ ; \tag{1}$$

a spatially flat universe corresponds to $k = 0$, a closed universe to $k = 1$, and an open universe to $k = -1$. (We set the speed of light $c = 1$ throughout.) In an expanding universe, the scale factor $a(t)$ is a growing function of time, which starts close to zero at the Big Bang about 13.7 billions years ago. The dynamics of the scale factor is given by the Friedmann equations,

$$\left(\frac{\dot a}{a}\right)^2 = \sum_i \frac{8\pi G}{3}\ \rho_i - \frac{k}{a^2}\ , \tag{2}$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}\sum_i(\rho_i + 3p_i)\ , \tag{3}$$

where the index i stands for any isotropic (comoving) perfect fluid. For radiation we have $p_{\rm r} = \rho_{\rm r}/3$, for dust $p_{\rm m} = 0$. For the recent accelerated expansion caused by some smooth dark energy component we would have $p_{\rm DE} = w_{\rm DE}\ \rho_{\rm DE}$, where $w_{\rm DE} < -1/3$ is still unknown and in many models time-dependent. ¿From (3) the expansion is typically decelerated, $\ddot a < 0$, unless at least one of the components satifies $\rho_i + 3p_i < 0$.

A space-independent scalar field $\phi(t)$ can be viewed as a comoving perfect fluid with

$$\rho_\phi = \frac{1}{2}\dot\phi^2 + V(\phi)\ , \tag{4}$$

$$p_\phi = \frac{1}{2}\dot\phi^2 - V(\phi)\ . \tag{5}$$

Hence, a scalar field $\phi(t)$ can induce an accelerated expansion provided

$$\dot\phi^2 < V(\phi)\ . \tag{6}$$

The field $\phi(t)$ driving the inflationary stage is called the inflaton and evolves according to the Klein-Gordon equation

$$\ddot\phi + 3H\dot\phi + \frac{dV}{d\phi} = 0\ , \tag{7}$$

which is the form taken by the conservation of energy for a perfect fluid defined by (4) and (5), and we have introduced the Hubble parameter $H \equiv \dot a/a$. In most inflationary models, the inflaton field $\phi(t)$ satisfies the slow-roll conditions $\ddot\phi \ll 3H\dot\phi$, and hence

$$3H\dot\phi \approx -\frac{dV}{d\phi}\ . \tag{8}$$

It is easy to show that the conditions for slow-roll to hold are

$$\dot H \ll 3H^2\ , \quad \frac{d^2V}{d\phi^2} \ll 9H^2\ , \tag{9}$$

in which case the condition (6) is amply satisfied so that accelerated expansion – inflation – takes place.

We conclude this brief summary on the background evolution during inflation by discussing the relative evolution of physical scales. The Hubble radius $R_{\rm H} \equiv H^{-1}$ defines an important scale in cosmology. If $a \propto t^p$, we have $R_{\rm H} \propto t$, and it is clear that $R_{\rm H}$ grows faster than a physical scale $\lambda \propto a$ during a decelerated expansion, which has $p < 1$. Hence physical scales greater than the Hubble radius, which we shall call "superhorizon" or "super-Hubble" scales, will eventually enter the Hubble radius, by which we mean that they will become smaller than $R_{\rm H}$: this is the situation in standard cosmology. This picture changes dramatically during inflation; to illustrate this we take a purely de Sitter stage, which is characterized by $H =$ constant and $a(t) \propto \exp(Ht)$. Now it is clear that physical scales inside the Hubble radius, which we shall call "subhorizon" or "sub-Hubble" scales will eventually become larger than the Hubble radius.

If a scale is said to cross the "horizon" 65 e-folds before the end of inflation, this means that at the end of inflation (where $t = t_{\rm e}$) one has $a = a_{\rm e} = {\rm e}^{65}a_k$ or $N_k = 65$ with

$$N_k = \frac{a_{\rm e}}{a_k}\ ; \tag{10}$$

here, $a_k \equiv a(t_k)$ if t_k is the "horizon-crossing" time of that particular scale with physical wavelength $(2\pi/k)a$. (Sometimes the factor 2π is omitted.) In a pure de Sitter stage this would mean that $H(t_{\rm e} - t_k) = 65$. If we can compute the present physical scale evolving from the Hubble radius during inflation, we know to which physical scale today a scale with given N_k corresponds. Depending on the details of the model, the Hubble radius *today* corresponds typically to a scale with $N_k \approx 65$. It can be shown that in slow-roll models N_k can be computed from the value $\phi(t_k)$ and that it depends on the potential $V(\phi)$.

In consistent inflationary scenarios, inflation is followed by a standard cosmic expansion during which scales that went outside $R_{\rm H}$ become again smaller than $R_{\rm H}$; they "re-enter the horizon". For a given scale, the number of e-folds between the first horizon crossing time t_k during inflation and the second horizon crossing time during the radiation or matter stage at $t = t_{k,\mathrm{f}}$ is given by the parameter r_k,

$$r_k \equiv \ln \frac{a(t_{k,\mathrm{f}})}{a_k} \equiv \ln \frac{a_{k,\mathrm{f}}}{a_k} \ . \tag{11}$$

We shall see in Sec. III that r_k coincides with the squeezing parameter for a quantum state[2]. For typical cosmological scales today, $r_k \sim 100$ and even larger. Physically this corresponds to an enormous expansion of the universe, while a given scale k was outside the Hubble radius. As we shall see below, the ensuing huge amount of squeezing for the quantum state plays a crucial role in the quantum-to-classical transition of inflationary quantum fluctuations. It also means that the quantum state originating from inflation is a very peculiar one.

B. Generation of perturbations

During an inflationary stage, quantum field fluctuations evolve according to the general principles of quantum field theory. Inflation is supposed to take place at an energy scale where space-time can be described as a classical curved space-time on which the quantum field fluctuations are defined. The inflaton fluctuations $\delta\phi(\mathbf{x}, t)$ can be treated as a massless scalar field. This is an excellent approximation when the inflaton field satisfies the slow-roll conditions (9) and it is even exact when we consider primordial gravitational waves.

It is convenient to consider the rescaled quantity $a\delta\phi \equiv y(\mathbf{x}, t)$ and to work with conformal time $\eta = \int dt/a(t)$; a prime will be used to denote a derivative with respect to η. The formalism presented here is exact for gravitational waves, but can be extended in a straightforward way to the primordial density perturbations.

The quantization of the real perturbation $y(\mathbf{x}, \eta)$ proceeds with the usual canonical quantization scheme. We start from the classical Hamiltonian describing the perturbations,

$$\begin{aligned} H &\equiv \int d^3\mathbf{x}\ \mathcal{H}(y, p, \partial_i y, \eta) \\ &= \frac{1}{2}\int d^3\mathbf{k}[p(\mathbf{k})p^*(\mathbf{k}) + k^2 y(\mathbf{k})y^*(\mathbf{k}) & (12) \\ &+ \frac{a'}{a}\left(y(\mathbf{k})p^*(\mathbf{k}) + p(\mathbf{k})y^*(\mathbf{k})\right)] \ , & (13) \end{aligned}$$

where p is the momentum conjugate to y,

$$p \equiv \frac{\partial \mathcal{L}(y, y')}{\partial y'} = y' - \frac{a'}{a} y \ . \tag{14}$$

In (13) we have introduced the (time-dependent) Fourier transform $y(\mathbf{k}, \eta)$ of the rescaled fluctuation $y(\mathbf{x}, \eta)$. (We sometimes keep the dependence on η.) In the Lagrangian formulation, it obeys the following classical equation of motion:

$$y''(\mathbf{k}, \eta) + \left(k^2 - \frac{a''}{a}\right) y(\mathbf{k}, \eta) = 0 \ . \tag{15}$$

Upon quantization, the Fourier transforms are promoted to operators on which we impose the canonical commutation relations,

$$[y(\mathbf{k}, \eta), p^\dagger(\mathbf{k}', \eta)] = \mathrm{i}\delta^{(3)}(\mathbf{k} - \mathbf{k}') \ . \tag{16}$$

(We set $\hbar = 1$.) We can write the Hamiltonian operator in the following way:

$$\begin{aligned} H &= \int \frac{d^3\mathbf{k}}{2}\left[k\left(a(\mathbf{k})a^\dagger(\mathbf{k}) + a^\dagger(-\mathbf{k})a(-\mathbf{k})\right) + \right. \\ &\quad \left. \mathrm{i}\frac{a'}{a}\left(a^\dagger(\mathbf{k})a^\dagger(-\mathbf{k}) - a(\mathbf{k})a(-\mathbf{k})\right)\right] \ . \end{aligned} \tag{17}$$

The time-dependent annihilation operators $a(\mathbf{k})$ (we often skip the argument η for conciseness) appearing in (17) are defined as usual,

$$a(\mathbf{k}) = \frac{1}{\sqrt{2}}\left(\sqrt{k}\ y(\mathbf{k}) + \frac{\mathrm{i}}{\sqrt{k}} p(\mathbf{k})\right) \ , \tag{18}$$

so that

$$\begin{aligned} y(\mathbf{k}) &= \frac{a(\mathbf{k}) + a^\dagger(-\mathbf{k})}{\sqrt{2k}} \ , & (19) \\ p(\mathbf{k}) &= -\mathrm{i}\sqrt{\frac{k}{2}}\left(a(\mathbf{k}) - a^\dagger(-\mathbf{k})\right) \ . & (20) \end{aligned}$$

It is easily seen from (16) that a and $a^\dagger$ satisfy the commutation relations

$$[a(\mathbf{k}, \eta), a^\dagger(\mathbf{k}', \eta)] = \delta^{(3)}(\mathbf{k} - \mathbf{k}') \ . \tag{21}$$

Let us consider the time evolution of these operators. ¿From the Hamiltonian (17) we get

$$\begin{pmatrix} a'(\mathbf{k}) \\ (a^\dagger(-\mathbf{k}))' \end{pmatrix} = k \begin{pmatrix} -\mathrm{i} & \frac{aH}{k} \\ \frac{aH}{k} & \mathrm{i} \end{pmatrix} \begin{pmatrix} a(\mathbf{k}) \\ a^\dagger(-\mathbf{k}) \end{pmatrix} . \tag{22}$$

The second piece of the Hamiltonian (17), which is proportional to a'/a, is responsible for a mixing between creation and annihilation operators. In the Heisenberg representation it corresponds to a Bogolubov transformation; physically it means that particles are produced in pairs with opposite momenta. For reasons that will become clear later, this phenomenon is called squeezing in the Schrödinger picture; the corresponding squeezing parameter r_k turns out to be given by the expression (11) above. ¿From (22) one can see that mixing of creation and annihilation operators is efficient when the off-diagonal terms dominate, in other words, on super-Hubble scales when $aH/k \gg 1$.

Using (20) and (22), one obtains after a little algebra,

$$y(\mathbf{k}, \eta) \equiv f_k(\eta)\ a_{\mathbf{k}} + f_k^*(\eta)\ a^\dagger_{-\mathbf{k}} \ , \tag{23}$$

where $a_{\mathbf{k}} \equiv a(\mathbf{k}, \eta_0)$, and the field modes f_k obey Equation (15) and satisfy $f_k(\eta_0) = 1/\sqrt{2k}$. At the initial time

η_0, the field modes are deep inside the Hubble radius. Equation (23) can be written in the suggestive way

$$y(\mathbf{k},\eta) = \sqrt{2k}\ f_{k1}(\eta)\ y_{\mathbf{k}} - \sqrt{\frac{2}{k}}\ f_{k2}(\eta)\ p_{\mathbf{k}}\ , \qquad (24)$$

where $y_{\mathbf{k}} \equiv y(\mathbf{k},\eta_0)$ and $p_{\mathbf{k}} \equiv p(\mathbf{k},\eta_0)$, $f_{k1} = \Re f_k$, $f_{k2} = \Im f_k$. We have in an analogous way momentum modes $g_k(\eta)$, with $g_k(\eta_0) = \sqrt{k/2}$,

$$p(\mathbf{k}) = \sqrt{\frac{2}{k}}\ g_{k1}(\eta)\ p_{\mathbf{k}} + \sqrt{2k}\ g_{k2}(\eta)\ y_{\mathbf{k}}\ . \qquad (25)$$

We shall now address the first step in understanding why and to which extent these quantum field modes appear classically.

III. QUANTUM-TO-CLASSICAL TRANSITION: THE PRAGMATIC VIEW

In the last section we have described the evolution of the quantum modes in the Heisenberg representation, in which operators evolve in time and quantum states do not. While the quantum-to-classical transition is in general formulated in the Schrödinger picture, for the inflationary perturbations the Heisenberg picture provides deep insight, too.

To see this, let us assume that there is a limit in which f_{k2} and g_{k1} (or f_{k1} and g_{k2}) vanish. Then it is clear from (24) that the non-commutativity of the operators $y_{\mathbf{k}}$ and $p_{\mathbf{k}}$ is no longer relevant. What is the physical meaning of such a limit? Let us consider a classical stochastic system where the dynamics is still described by equations of the form (24), but with now $y(\mathbf{k},\eta_0)$ and $p(\mathbf{k},\eta_0)$ representing random initial values (c-numbers). If f_{k2} and g_{k1} vanish, we get

$$p(\mathbf{k},\eta) \equiv p_{\rm cl}(y(\mathbf{k},\eta)) = \frac{g_{k2}}{f_{k1}}\ y(\mathbf{k},\eta)\ . \qquad (26)$$

This is true for the quantum system (in the operator sense) and for the classical stochastic system (in the c-number sense). Therefore, for a given realization of the perturbation $y(\mathbf{k},\eta)$, the corresponding momentum $p_{\rm cl}(\mathbf{k},\eta)$ is fixed and equal to the classical momentum corresponding to this value $y(\mathbf{k},\eta)$. Then the quantum system is effectively equivalent to the classical random system, which is an ensemble of classical trajectories with a certain probability associated to each of them[3].

This is, in fact, what happens for the primordial fluctuations. The field modes obey (15), and this equation has, on super-Hubble scales, solutions that become dominant and solutions that become negligible (so-called "growing" and "decaying" modes). Eventually the decaying mode can be neglected and one in left with the growing mode. It turns out that f_{k2} and g_{k1} are decaying modes, and one is left with (26).

From the Heisenberg representation it follows that the operational equivalence with the classical stochastic system does not depend on the initial state; this was indeed shown explicitly for a wide class of initial states (and extended to some gauge-invariant quantities)[4].

We now look at the problem in the Schrödinger representation where the state evolves in time, while the operators are fixed. The initial quantum state of the perturbations is the vacuum state $|0,\eta_0\rangle$ satisfying

$$a_{\mathbf{k}}|0,\eta_0\rangle = 0 \qquad\qquad \forall\mathbf{k}\ . \qquad (27)$$

At later times, due to the creation of particles, the time-evolved state is annihilated by a more complicated operator,

$$\left\{y_{\mathbf{k}} + \mathrm{i}\gamma_k^{-1}(\eta)p_{\mathbf{k}}\right\}|0,\eta\rangle = 0\ . \qquad (28)$$

The corresponding (Gaussian) wave function reads

$$\begin{aligned}\Psi[y_{\mathbf{k}},y^*_{\mathbf{k}},\eta] &= \frac{1}{\sqrt{\pi|f_k|^2}}\exp\left(-\frac{|y_{\mathbf{k}}|^2}{2|f_k|^2}\{1-\mathrm{i}2F(k)\}\right)\\ &\equiv \left(\frac{2\Omega_{\rm R}(\eta)}{\pi}\right)^{1/4}\exp\left(-[\Omega_{\rm R}(\eta)+\mathrm{i}\Omega_{\rm I}(\eta)]|y_{\mathbf{k}}|^2\right)\ . \qquad (29)\end{aligned}$$

In (28,29), we have

$$\begin{aligned}\gamma_k &= \frac{1}{2|f_k|^2}[1-2\mathrm{i}F(k)]\ ,\\ F(k) &= \Im f_k^* g_k\ = f_{k1}g_{k2} - f_{k2}g_{k1}\ . \qquad (30)\end{aligned}$$

At the initial time $\eta = \eta_0$, $\gamma_k(\eta_0) = k$, and hence $F(k) = 0$; in other words, we have a minimum uncertainty wave function. This is no longer so later, as $|F(k)|$ becomes very large; the probabilities, however, remain Gaussian. Another way to exhibit the physical meaning of our state is to consider the Wigner function, W, which can be considered as a kind of quasi-probability density in phase space. For Gaussian wave functions, W has the property to be positive definite. For the wave function (29) one obtains

$$W \stackrel{|r_k|\to\infty}{=\!=\!\longrightarrow} |\Psi|^2\ \delta^{(2)}\ \left(p_{\mathbf{k}} - p_{\rm cl}(y_{\mathbf{k}})\right)\ . \qquad (31)$$

The dynamics of the fluctuations leads to the large-squeezing limit $|r_k| \to \infty$. One gets a highly elongated ellipse whose large axis is oriented along the line $p_{\mathbf{k}} = p_{\rm cl}(y_{\mathbf{k}})$ and whose width becomes negligible. This is a direct vizualisation of the classical stochastic behaviour of our system: the variable $y_{\mathbf{k}}$ can take any value with corresponding probability $|\Psi|^2$, while $p_{\mathbf{k}}$ takes the corresponding value $p_{\mathbf{k}} = p_{\rm cl}(y_{\mathbf{k}})$. Instead of being essentially located in phase space around one physical trajectory, as for coherent states, the system behaves as if it followed an infinite number of classical trajectories with a definite probability to be on each of them. Interestingly, an analogous situation happens for a free non-relativistic particle[5] possessing an initial Gaussian minimal uncertainty wavefunction. As is well known, $F \propto t$ and becomes very large. At very late times, the position does no longer depend on the initial position,

$$x(t) \simeq \frac{p_0}{m}t\ . \qquad (32)$$

We get an equivalence with an ensemble of classical particles obeying (32), where p_0 is a random variable with probability $P(p_0) = |\Psi|^2(p_0)$. This illustrates the kind of classicality we are dealing with. Moreover, when (32) holds, position operators at different times approximately commute (which, in quantum-optical language, corresponds to a quantum-nondemolition situation).

Using the canonical commutation relations, the quantum coherence between the growing and decaying mode can be expressed as

$$f_{k1}g_{k1} + f_{k2}g_{k2} = \frac{1}{2} \ . \tag{33}$$

Clearly, when f_{k2}, g_{k1} are unobservable, this coherence becomes unobservable as well. This is the case when the decaying mode is so small that we have no access to it in observations. For the ratio of the growing to the decaying mode one has

$$\frac{f_{k2}}{f_{k1}} \propto e^{-2|r_k|} \ , \tag{34}$$

which is why a large squeezing parameter r_k in the Schrödinger picture implies a vanishing decaying mode in the Heisenberg representation. The width of the Wigner function is given by

$$\langle (p_{\mathbf{k}} - p_{\mathrm{cl}}(y_{\mathbf{k}}))^2 \rangle = g_{k1}^2 \ , \tag{35}$$

which becomes unobservable like the decaying mode. A further consequence is that the typical phase-space volume occupied by the system becomes negligible, too.

Let us take the concrete and important example of a perturbation on de Sitter space $a \propto \mathrm{e}^{Ht}$, with H being constant. The exact solution of (15) with the correct initial condition (ground state for initial sub-Hubble modes) then reads up to an unimportant *constant* phase factor

$$f_k = \frac{-\mathrm{i}}{\sqrt{2k}} \, \mathrm{e}^{-\mathrm{i}k\eta} \big(1 - \frac{\mathrm{i}}{k\eta}\big) \ , \tag{36}$$

$$g_k = -\mathrm{i} \sqrt{\frac{k}{2}} \mathrm{e}^{-\mathrm{i}k\eta} \ , \quad \eta \equiv -\frac{1}{aH} < 0 \ . \tag{37}$$

Modes initially inside the Hubble radius become much larger than the Hubble radius during inflation solely as a result of their dynamics to satisfy $k\eta \ll 1$: here we have the limit mentioned above! This can be shown also to correspond to the large-squeezing limit. Actually, this is a particular case of the general situation when an equation like (15) has a growing-mode solution and a decaying-mode solution. Here the decaying mode becomes vanishingly small; when it is neglected we are in the limit of a random stochastic process. Perturbations are then given by

$$\delta\phi(\mathbf{k}, \eta) = \frac{H}{\sqrt{2k^3}} \ e_{\mathbf{k}} \ . \tag{38}$$

We have set here $\sqrt{2k} \ y_{\mathbf{k}} = e_{\mathbf{k}}$, which assumes the role of a classical Gaussian random variable with unit variance. From (38) we see that the perturbations tend to a constant value (they become "frozen"). One should realize that the true reason for the quantum-to-classical transition in the sense discussed here is that the decaying mode becomes vanishingly small. Primordial gravitational waves follow exactly the behaviour (38) (up to some factor)[6], but after re-entering the Hubble radius they will start oscillating. They retain their classical appearance because the decaying mode (which oscillates as well by then!) is negligible[3].

IV. OBSERVATIONAL PREDICTIONS

The perturbations produced during inflation have remarkable properties which can be confronted with observations. This confrontation makes essential use of the effective classical behaviour discussed in the last section.

Primordial inflaton fluctuations generate a primordial Newtonian potential and the corresponding energy-density fluctuations $\delta\rho$. A central quantity is the power spectrum, $P(k)$, of the quantity $\delta \equiv \delta\rho/\rho$,

$$\langle \delta(\mathbf{k}) \ \delta^*(\mathbf{k}') \rangle = P(k) \ \delta^{(3)}(\mathbf{k} - \mathbf{k}') \ . \tag{39}$$

When the statistical properties are isotropic, the power spectrum depends only on $k \equiv |\mathbf{k}|$. It can be shown that the power spectrum is the Fourier transform of the correlation function (in space), and it can be defined for any quantity. Deep in the matter-dominated stage, $P(k)$ has the following expression on "super-horizon" scales in slow-roll single-field inflation,

$$P(k) = \frac{1024}{75} \pi^3 G^3 \left(\frac{V^3}{V'^2} \right)_{t_k} (aH)^{-4} \ k, \tag{40}$$

where V' is the derivative of the inflaton potential with respect to the inflaton ϕ, and the fraction has to be evaluated at the Hubble-radius crossing time $k = a(t_k)H(t_k)$ *during* inflation. Because of the quasi-exponential inflationary expansion, it depends very weakly on k. Neglecting this dependence, we get

$$P(k) \propto k \ , \tag{41}$$

which is the scale-invariant "Harrison–Zeldovich" spectrum that plays a crucial role in these investigations. This spectrum is called scale-invariant for the following reason: if we compute the r.m.s. relative mass fluctuations $\langle (\delta M/M)^2 \rangle$ at the time t_k when a scale eventually re-enters the Hubble radius, the same value is obtained for all scales.

Using the expansion (23) and the commutation relations (21), it is straightforward to show that

$$\langle \delta\phi^2 \rangle = \frac{1}{2\pi^2} \int_0^\infty dk \ k^2 \ |\delta\phi_k(\eta)|^2 \ , \tag{42}$$

with $f_k(\eta) = a \ \delta\phi_k(\eta)$. This means that the power spectrum of $\delta\phi$ is just given by $|\delta\phi_k(\eta)|^2$. However, the average on the left is a quantum average; it is only by virtue of the quantum-to-classical transition mentioned above that we can consider $|\delta\phi_k(\eta)|^2$ as the power spectrum of a classical random variable, whose time evolution is consistent with probabilities conserved along classical trajectories. In the opposite case this would be impossible due to quantum interferences. We note also the result in the limit (38), which gives

$$\frac{d\langle\delta\phi^2\rangle}{d\ln k} = \left(\frac{H}{2\pi}\right)^2 , \tag{43}$$

where the derivative is with respect to some cut-off value.

Primordial fluctuations leave their imprint on the CMB and this provides the best constraint on their properties and on the inflationary models in which they were presumably produced. While the CMB is remarkably homogeneous with a black body spectrum, perturbations induce very tiny inhomegeneities of the order 10^{-5}. In this regime, linear perturbation theory is very accurate so that precise predictions can be made. The measurement of the temperature anisotropies angular power spectrum, the C_ℓ's,

$$C_\ell = \langle|a_{lm}|^2\rangle , \quad \frac{\Delta T}{T}(\vartheta,\varphi) = \sum_{l,m} a_{lm}\, Y_{lm} , \tag{44}$$

(which are in the isotropic case independent of m) will culminate with the Planck satellite (ESA). The exquisite data we have thus far, in particular those collected by the WMAP collaboration (NASA), show excellent agreement with a flat universe and adiabatic perturbations[7,8]. Such perturbations respect the equation of state of the background; for the baryon–photon plasma this is when $\frac{\delta T}{T} = \frac{1}{3}\frac{\delta n_B}{n_B}$, where n is the baryon number density. This is a natural outcome of single-field inflation.

Before decoupling, the baryon–photon plasma is tightly coupled and its density oscillates on scales inside the Hubble radius, yielding oscillations similar to pressure waves. These are often called acoustic oscillations. The location of the first (Doppler) peak gives roughly the angular scale of the Hubble radius at decoupling and is consistent with a flat universe. The pattern of the angular power spectrum is in agreement with primordial adiabatic fluctuations. After decoupling, the baryons retain the primordially induced acoustic "Sakharov" oscillations, the baryonic acoustic oscillations (BAO); these were detected in the galaxy power spectrum and are presently used in order to constrain dark energy models.

To parametrize the departure from scale invariance, one introduces the spectral index n with $P(k) \propto k^n$. Latest CMB data constrain n to be very close, but slightly lower than one[7]. Finally we see no clear evidence for non-Gaussianity in the statistics of the perturbations. All these data are in surprisingly good agreement with the simplest single-field slow-roll inflationary models (see e.g.[9]).

Let us return in more detail to the acoustic oscillations. They arise because of the standing-wave behaviour of the perturbations inside the Hubble radius. There are always two modes that are solutions to the equations and they will both oscillate. One of the modes matches the growing (dominant) mode, and the other the decaying (subdominant) mode. For modes sufficiently long outside the Hubble radius, the decaying mode disappears and the growing mode will match the corresponding oscillating mode inside the Hubble radius. At decoupling, each mode has a given oscillation phase, and this gives rise to the acoustic oscillations seen in the C_ℓ's. If we had a way to generate classical perturbations that would evolve outside the Hubble radius for very long, just the same would be true. If these perturbations had random initial conditions, obeying the same statistics as our initially quantum fluctuations, both systems would be indistinguishable. Hence the presence of acoustic oscillations is in no way connected to the quantum nature of the perturbations but rather to their primordial origin. But the quantum-to-classical transition can only take place in a system where the decaying mode is negligible enough so that acoustic oscillations *do* arise. It is interesting that a similar standing-wave behaviour is present in the primordial stochastic gravitational waves background produced during inflation. Unfortunately, to detect it in a direct detection experiment today would require a resolution in frequency of about 10^{-18} Hz,[3] clearly beyond present or foreseeable capabilities. The same property yields also small superimposed oscillations in the power spectra of the CMB temperature anisotropy and polarization. This is similar to the acoustic oscillations but with a period approximately twice as small (solely due to the difference between the light velocity and the sound velocity in the baryon–photon plasma at the recombination time)[3]. Their observation is very difficult but not hopeless if the parameter characterizing the tensor-to-scalar ratio in the CMB temperature anisotropy is not too small, see[10] for detailed estimates of the CMB polarization B-mode produced by primordial gravitational waves only.

We finally mention that calculations done for the creation of matter by parametric resonance after inflation use the description of perturbations in terms of classical stochastic fields. All the predictions mentioned above and which were confirmed by observations are done in the closed-system approach, that is, by taking the perturbations as an isolated system. Similar results were obtained in various disguise by several authors[11,12,13] and even extended beyond the linear regime[14]. In this approach the system becomes indistinguishable, in an operational sense, from a classical stochastic system solely by virtue of its peculiar inflationary dynamics.

From a purely pragmatic point of view, the closed-system approach is sufficient. In astrophysical observations one measures certain classical correlation functions for which the above line of thought shows that they are indistinguishable from the fundamental quantum expectation values. Still, in the next section we shall go beyond the closed-system approach by taking into account the interaction of the modes with other, "environmental", degrees of freedom. This has several reasons. First, the environment-induced decoherence process is generally invoked in order to explain the appearance of classical behaviour in quantum theory[15]. Second, since an environment is expected to be present anyway, it is important to consider whether it does not spoil the successful predictions from the closed-system approach. It should, in particular, not erase the acoustic oscillations. Moreover, invoking large non-linear effects might irremediably modify the CMB angular power spectrum and induce large non-Gaussianity. Finally, there is the question about the entropy of the perturbations which by definition cannot be addressed inside the closed-system approach.

We shall see that these questions and problems can be successfully dealt with without spoiling the successful predictions of the closed-system approach including the quantum-to-classical transition in the pragmatic approach adopted in this section.

V. QUANTUM-TO-CLASSICAL TRANSITION: DECOHERENCE

A. Decoherence and pointer basis

In the last section we have described the primordial fluctuations in cosmology by a collection of independent quantum states labelled by the wave number k. Since no interaction between different k or between the fluctuations and other fields have been considered, we deal with a pure quantum state for each k. The initial condition for each quantum state is the harmonic-oscillator ground state with respect to k. During inflation, modes with wavelengths larger than the Hubble scale H^{-1} assume a squeezed Gaussian state. We focussed attention on the modes far outside the Hubble scale, which experience an enormous squeezing. For these highly-squeezed modes, which are the ones relevant for cosmological observations, all expectation values containing the field-amplitudes or their momenta are indistinguishable from classical stochastic averages[3]. It is this approximate coincidence between quantum and classical expectation which is the basis of the pragmatic approach to the quantum-to-classical transition discussed above for the primordial fluctuations.

One can, however, adopt a more fundamental point of view. It is far from realistic to assume that a primordial fluctuation with wave number k is exactly isolated. We must take into account its interaction with other degrees of freedom (called the 'environment' for simplicity). The main reason is the following. As one knows from standard quantum theory, even a tiny interaction with other degrees of freedom can become important, in the sense that an entanglement of a system with its environment can form even without direct disturbance of the system. If the environmental degrees of freedom are inaccessible to observations (as they usually are), the ensuing entanglement with the system leads to *decoherence* – interference terms can no longer be observed at the system itself and the system *appears* classical[15]. This is the fundamental origin of the quantum-to-classical transition. The phenomenon of decoherence is by now theoretically well understood and has been experimentally tested with high precision[15,16,17]. Decoherence leads to an *apparent ensemble* of wave packets for the observable with respect to which the interferences vanish. A paradigmatic example is the localization of a quantum particle due to scattering with photons, air molecules, or other particles[15,17,22]. There the position basis of the particle is the approximate basis distinguished by the scattering process. The basis distinguished by the environment is generally called the *pointer basis*; the corresponding observable is called pointer observable. Interferences between different members of the pointer basis are suppressed by the decohering influence of the environment.

One would expect, therefore, that decoherence is of crucial importance for the primordial fluctuations, too. This expectation is, moreover, supported by the fact that the system by itself evolves into a highly squeezed state in which squeezing is in the field momentum and broadening is in the field amplitude (corresponding to the position variable in quantum mechanics): one knows from quantum theory that highly squeezed states are extremely sensitive to any environment[15]. This is the reason why they are so difficult to generate in the laboratory – it is very hard to isolate them from any environment. In view of their huge squeezing, this argument should apply to the cosmological fluctuations *a fortiori*.

But could it be imaginable that the cosmological fluctuations, in contrast to a typical quantum-mechanical situation, are indeed strictly isolated? The answer is definitely no.

Firstly, in any fundamental theory (such as string theory) there is an abundance of different fields with different interactions. Among them it will not be difficult to find appropriate candidates for environmental fields generating decoherence for the primordial fluctuations.

Secondly, even if one assumes to have no such fields, there are two processes which cannot be neglected. The first one is the interaction between modes with different k; recall that the full theory is non-linear and that, therefore, the various modes cannot be treated independently of each other. Such non-linear interactions concern both the interaction with the modes of the inflaton and the perturbations of the metric (containing, in particular, gravitational waves).

The second process is the entanglement of the modes' quantum state between different *spatial regions*: even if the modes are independent in k-space, the Gaussian wave functions for the amplitudes in real space are highly correlated over spacelike regions (as in the Einstein–Podolsky–Rosen situation). This leads, in particular, to an entanglement between the regions inside and outside the Hubble radius. Famous non-cosmological examples are the Hawking and the Unruh effects, where the thermal appearance of the corresponding radiation can be understood from the entanglement between inside and outside the event horizons and the tracing out of the correlations into the horizon[18]. Even for space-like surfaces which stay outside the horizon, the thermal nature of Hawking and Unruh radiation can be understood from the entanglement with other fields, leading to decoherence[19].

The process of decoherence is, moreover, needed to justify the results from the isolated (closed) system in the first place. Even if the classical and quantum expectation values are indistinguishable, the presence of a pure state means that one has a quantum superposition of all possible field amplitudes, *not* an ensemble of stochastically distributed classical values. This situation is similar to Schrödinger's cat. In the pragmatic point of view of Sec. III, the approximate coincidence of the expectation values suffices. Such a coincidence is, however, not sufficient for a realistic interpretation. Only decoherence can eventually justify the pragmatic point of view in that it leads to an apparent ensemble of wave packets for the system variables itself (which, in our case, are the field amplitudes). The insufficience of approximately equal classical and quantum expectation values for a fundamental interpretation has recently been clearly emphasized in a different context (the quantum mechanics of classically chaotic systems) by Schlosshauer[20]. In the presence of a pure state one can always find an observable for which no classical counterpart exists, that is, for which the comparison of quantum and classical expectation values is meaningless.

The quantum-to-classical transition happens for the highly-squeezed modes whose wavelengths exceed the Hubble scale. It is for these modes where environmental decoherence is most efficient[21]. How can this happen? Would one not expect that no causal interaction can occur on scales larger than the Hubble scale? This is true only for a direct disturbance of the system. But the crucial point is that quantum entanglement can form without direct disturbance. And this is all one needs for decoherence! In the context of the quantum measurement process, the sole formation of entanglement is referred to as an 'ideal measurement' or a 'quantum non-demolition measurement': the system remains undisturbed, but the environment is affected through the formation of entanglement. The general mechanism is as follows[15].

Consider a quantum system which is initially in the state $|n\rangle$ and a 'measurement device' (here: the environment) which is in some initial state $|\Phi_0\rangle$. (We assume that $|n\rangle$ belongs to a set of eigenstates of a system observable.) The evolution according to the Schrödinger equation is in the special case of an 'ideal measurement' given by

$$|n\rangle|\Phi_0\rangle \xrightarrow{t} \exp\left(-\mathrm{i}H_{\mathrm{int}}t\right)|n\rangle|\Phi_0\rangle = |n\rangle|\Phi_n(t)\rangle \ , \quad (45)$$

where H_{int} denotes the interaction Hamiltonian (assumed here to dominate over the free Hamiltonians) which correlates the system state with its environment without changing the system state.

In the general case, the quantum system can be in a superposition of different eigenstates of the system observable. Then, due to the linearity of the time evolution, an initial product state with $|\Phi_0\rangle$ develops into an entangled state of system plus apparatus,

$$\left(\sum_n c_n|n\rangle\right)|\Phi_0\rangle \xrightarrow{t} \sum_n c_n|n\rangle|\Phi_n(t)\rangle \ . \quad (46)$$

But this is a highly non-classical state! Since the environmental states $\{|\Phi_n\rangle\}$ are not accessible, they have to be traced out from the full quantum state. One thereby arrives at the reduced density matrix ρ_{S} which contains all the information that is available at the system itself. Since the environmental states $\{|\Phi_n\rangle\}$ can be assumed as being approximately orthogonal (otherwise they would not be able to serve as a 'measurement device'), the reduced density matrix is of the form

$$\rho_{\mathrm{S}} \approx \sum_n |c_n|^2 |n\rangle\langle n|, \quad (47)$$

that is, it assumes the form of an *approximate ensemble* for the various system states $|n\rangle$, each of which occurs with probability $|c_n|^2$.

In our case, the cosmological fluctuations represent the system to be decohered. The environmental states $\{|\Phi_n\rangle\}$ can be other fields or inaccessible parts of the fluctuations themselves (see below). The system states $|n\rangle$ are given by the field-amplitude states $|y_{\mathbf{k}}\rangle$. The interaction with the environment can, in the ideal-measurement case, be described by the multiplication of an initial density matrix $\rho_0(y,y')$ with a Gaussian factor in $y-y'$ (omitting here and in the following the index $\mathbf{k}$ in $y_{\mathbf{k}}$),

$$\rho_0(y,y') \longrightarrow \rho_\xi(y,y') = \rho_0(y,y')\exp\left(-\frac{\xi}{2}(y-y')^2\right) . \quad (48)$$

Here, the parameter ξ encodes the details of the interaction between the modes and their environment. Given a specific model with a specific interaction, ξ can be calculated. The special decoherence process (47) is typical for the description of localization in quantum mechanics[15,17,22].

One recognizes from (48) that interferences between different values of the field amplitude y have been suppressed by interaction with the environment. This is decoherence. So far we have just assumed without derivation that $|y\rangle$ is the pointer basis, that is, the relevant robust system basis which is distinguished by the environment. This must, of course, be justified. A detailed derivation for the field-amplitude basis to be the pointer basis has been presented in[21] and[25]. We review here the main arguments and refer the reader to these references for more details.

According to the classical equations, for modes with very large wavelength one has $y \propto a$, that is, the physical fluctuations $\delta\phi$ are approximately constant ('frozen'). In the Heisenberg picture of the quantum theory, this means that the operator $\widehat{\delta\phi}$ approximately commutes with the Hamiltonian. Now comes the crucial point. Additional (environmental) fields coupling with the cosmological fluctuations are expected to couple field amplitudes, not canonical momenta of field amplitudes; that is, the coupling is expected to involve $\widehat{\delta\phi}$, *not* its momentum. Consequently, the fluctuations $\widehat{\delta\phi}$ commute with the *whole* Hamiltonian of system plus environment. Such a variable is a pointer observable par excellence[15,16,17]. It is stable (robust) in time because of this commutativity which holds for the wavelengths much bigger than the Hubble scale. The phenomenological expectation (48) is thus fully justified. One must keep in mind, though, that $\widehat{\delta\phi}$ is only an approximate pointer observable: although the non-diagonal terms in (48) become exponentially suppressed, they never vanish exactly, as would be the case if the $\widehat{\delta\phi}$ were the exact pointer observable. In fact, the reduced density matrix can be decomposed into narrow Gaussians in $\delta\phi$-space. The whole situation is in strong analogy to the localization of a massive particle by scattering with the environment[15,17,22].

The approximate commutativity of $\widehat{\delta\phi}$ with the full Hamiltonian means in particular that the kinetic term, that is, the p^2-term, of the system becomes irrelevant in the large-squeezing limit. If this term were relevant (as it is for modes with smaller wavelength), the pointer basis would not be the field-amplitude basis, but the coherent-state basis[25]. But this is not the case here. The coherent-state basis is, in particular, unstable under the time evolution.

So far we have restricted our attention to a special initial state: the vacuum state. This is, however, not necessary. In[25] we have presented a formalism that is general enough to encompass a wide range of initial states and interactions. A central role in this formalism is played by a master equation for the reduced density matrix, which is of the Lindblad form. More concretely, the density matrix is assumed to satisfy[15]

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\mathrm{i}[\hat{H},\hat{\rho}] + \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}\hat{L}^\dagger\hat{L}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}^\dagger\hat{L} \ , \tag{49}$$

where $\hat{L}$ is the Lindblad operator. Most of the particular models discussed in the literature lead to a master equation of this form. It is thus of interest to study this equation as general as possible. We have assumed that the Lindblad operator is linear in our variables p and y, but kept it general otherwise. The Hamiltonian $\hat{H}$ is given by the expression (17).

The results of our discussion in[25] can be summarized as follows. It turns out that the behaviour of the master equation is qualitatively different for modes outside the Hubble radius (as is the case here) and the modes inside. The decoherence time t_{d} for the modes with wavelengths much bigger than the Hubble radius is during inflation of the order

$$t_{\mathrm{d}} \sim H_{\mathrm{I}}^{-1} \ln \frac{H_{\mathrm{I}}^{-1}}{t_0}, \tag{50}$$

where H_{I} is the (approximately constant) Hubble parameter during inflation, and t_0 is a typical time characteristic for the details of the interaction. We emphasize that (50) is approximately independent of these details. It is basically given by the Hubble time, with the details only entering logarithmically. The time t_{d} also gives the timescale for the Wigner function to become positive. The reduced density matrix can then be decomposed into an apparent ensemble of narrow Gaussians for the values of the field amplitude, cf.[26] for a general discussion. For the large-wavelength modes in the radiation-dominated phase one obtains instead

$$t_{\mathrm{d}} \sim \frac{H_{\mathrm{I}} t_{\mathrm{L}}^2}{2}, \tag{51}$$

where t_{L} depends again on the details of the interaction. One has now a more sensitive dependence on the interaction. Moreover, for $H_{\mathrm{I}} t_{\mathrm{L}} \gg 1$ one has a much longer decoherence time than during inflation. This means that, depending on the interaction, decoherence can be much less efficient than during inflation.

For modes smaller than the Hubble scale, the situation is very different[25]. Taking as a representative example a photon bath as the environment (realized e.g. by the CMB), the decoherence time is independent of the Hubble parameter and strongly dependent on the coupling to the bath. Dissipation now becomes the dominant source of influence, in contrast to the case of the super-Hubble modes for which only entanglement occurs.

Decoherence is often connected with symmetry breaking[15], see also[27], section 6.1. This is also the case here. The initial de Sitter-invariant vacuum state for the fluctuations is highly symmetric. But the observed classical fluctuations are certainly non-symmetric. This can easily be understood and does not require new physics (as e.g. demanded in[28]). The initial vacuum state develops into a squeezed vacuum, which can be understood as a superposition of different field-amplitude eigenstates. Decoherence then makes this indistinguishable from an ensemble of (approximate) field-amplitude eigenstates, each of which is highly inhomogeneous. The situation resembles the case of spontaneous symmetry breaking in field theory, where the symmetric initial state evolves into a superposition of 'false vacua'. After decoherence one is left with an apparent ensemble of different false vacua, one of which corresponds to our observed world.

B. Entropy

In Sec. III the primordial fluctuations were treated as isolated and thus described by a pure (squeezed) state. Consequently, they possess zero entropy: all information is contained in the system itself. But as we have seen, the primordial fluctuations are an *open* quantum system; they are entangled with their environment. Because of this entanglement, the fluctuations are described by the reduced density matrix (48). They thus possess positive entropy because the information about the correlations with the environment are unavailable in the system itself. The local entropy is calculated from the standard von Neumann formula,

$$S = -\mathrm{tr}(\rho_\xi \ln \rho_\xi) \ , \tag{52}$$

where ρ_ξ is given in (48), and where we have set $k_{\mathrm{B}} = 1$. Considering one (real) mode with wave number k, the maximal entropy, S_{max}, would be $2r_k$, where r_k is again the squeezing parameter[29] (we skip again the index k in the following). We have calculated and discussed the entropy for the fluctuations in[23,25]. To display the result, it is convenient to introduce the dimensionless parameter $\chi = \xi/\Omega_{\mathrm{R}}$, where Ω_{R} is the width of the Gaussian (29); it controls the strength of decoherence. (In the case of pure exponential inflation one has $\chi = \xi(1+4\sinh^2 r)/k$.) Inserting (48) into (52), one gets the explicit expression[25]

$$\begin{aligned} S &= -\ln\frac{2}{\sqrt{1+\chi}+1} - \frac{1}{2}\left(\sqrt{1+\chi}-1\right)\ln\frac{\sqrt{1+\chi}-1}{\sqrt{1+\chi}+1} \\ &= \ln\frac{1}{2}\sqrt{\chi} - \sqrt{1+\chi}\ln\frac{\sqrt{1+\chi}-1}{\sqrt{\chi}} \ . \end{aligned} \tag{53}$$

One recognizes that the entropy vanishes for $\xi \to 0$, as it must for a pure state. In the limit $\chi \gg 1$ (large decoherence) one gets

$$S = 1 - \ln 2 + \frac{\ln\chi}{2} + \mathcal{O}(\chi^{-1/2}) \ . \tag{54}$$

This asymptotic value is readily attained.

As we have emphasized above, modes with wavelength bigger than the Hubble scale can only experience pure entanglement, not direct disturbance. In such a case the entropy obeys the bound

$$S < \frac{S_{\mathrm{max}}}{2} = r \ . \tag{55}$$

The same bound follows from the general discussion of the Lindblad equation[25]. It can also be interpreted in the following way[25]: in spite of decoherence, some squeezing compared to the vacuum state (which has $\Omega_{\mathrm{R}} = k$) remains. In the language of the Wigner function it means that the Wigner ellipse is not smeared out to become a circle, but still exhibits an elongated and a squeezed part. And this has important consequences for observation! If the bound (55) were violated, there would no longer be any coherences between the field amplitude and the momentum and, consequently, no coherences in the coupled baryon–photon plasma (Sec. IV). There would then not be any acoustic peaks in the anisotropy spectrum of the CMB – in contrast to observation! The fundamental questions of the quantum-to-classical transition have thus observational relevance.

The upper bound $S_{\mathrm{max}}/2$ corresponds to the case when the pointer basis is the exact field-amplitude basis. (For $S = S_{\mathrm{max}}$, the pointer basis would be the particle-number basis.) As our pointer basis consists of narrow packets in field amplitudes, the entropy of the fluctuations approaches the upper bound asymptotically.

The existence of the bound (55) shows, again, how peculiar the case of fluctuations in an inflationary universe is. According to a theorem by Page[30] (see also[31]), if a total quantum system with dimension mn is in a random pure state, the average entropy of a subsystem of dimension $m \leq n$ is almost maximal. But this is not the case for our system: the situation for the fluctuations during inflation *is* very special, and their entropy cannot exceed half of the maximal entropy, which leaves enough information for the formation of the acoustic peaks.

Our results for the entropy in[23] and[25] also yield the following simple formula for the entropy production during inflation:

$$\dot{S} \approx \dot{r} \approx H_{\mathrm{I}} \ . \tag{56}$$

For chaotic systems, the entropy production rate is proportional to the Lyapunov parameter. This would correspond in our case to the Hubble parameter H_{I}. However, our system is not chaotic, but only classically unstable, so the analogy is not complete.

Using (50), one can find the amount of entropy produced after the decoherence time t_{d},

$$S \sim H_{\mathrm{I}} t_{\mathrm{d}} \sim \ln\frac{H_{\mathrm{I}}^{-1}}{t_0} \ . \tag{57}$$

In the radiation-dominated phase following inflation, a relation similar to (56) holds, with H_{I} replaced by the Hubble parameter $H \propto t^{-1}$. The entropy thus only increases logarithmically in time, not linearly as in inflation.

C. Specific models

So far, we have kept the discussion as general as possible. We have reviewed the arguments which lead to the result that cosmological fluctuations appear like a classical ensemble of field amplitudes. Necessary requirements are the inflationary expansion of the universe and the focus on modes that are highly squeezed. An interaction with some environment is needed, but the details of it are unimportant. Still, it is of interest to discuss specific examples for such interactions. Our paper[25] gives an extended list of references; here we shall restrict ourselves to some recent examples.

The purely spatial entanglement between the modes inside the Hubble scale and outside the Hubble scale was discussed in[32], see also[33]. It was shown there that this entanglement is, by itself, sufficient to produce the desired decoherence. This is analogous to the black-hole case where the decoherence from the tracing out of the modes behind the horizon leads to the thermal radiation of the Hawking effect[18,19]. The authors of[32] also showed that the entropy scales with the volume inside the Hubble scale and satisfies an upper bound of $S \approx r$ per mode, which coincides with the upper bound (55) discussed above. It is thus not in conflict with the observed acoustic peaks in the cosmic microwave background.

Instead of pure spatial entanglement one can consider the entanglement of our strongly squeezed super-Hubble modes with sub-Hubble modes (which then play the role of the environment). This was discussed, for example, in[34]. The authors take the short-wavelength modes to be in their ground states and find that decoherence is not sufficient during inflation. This happens because vacuum states are usually ineffective to lead to decoherence[15]. Our arguments above and in[25] can thus only be applied to this model if at least some modes are not in their ground states. But such modes can be found: one can interpret the fluctuations with wavelengths $\lambda \gtrsim H_{\rm I}^{-1}$ as an appropriate environment; they assume a role intermediate between ground state and state with large squeezing. Ideas similar to the ones in[34] have been pursued in[35,36], and elsewhere, with results that are consistent with our general discussion above. A variant of this system-environment split is presented in[37] using a two-field model of inflation. There, the system consists of curvature perturbations, and the environment consists of isocurvature modes. Finally, another possible source of sub-Hubble modes being in non-vacuum states is the *secondary* gravitational wave background ("foreground" in astronomical terminology) emitted by matter after the end of inflation[23].

VI. CONCLUSION

Inflation is a robust scenario which gives an elegant solution to some oustanding problems of Big-Bang cosmology, and its predictions are in agreement with present observations, in particular the accurate CMB anisotropy data. It is gratifying that this scenario offers also the possibility to deal with such fundamental and subtle questions as to why quantum perturbations produced in the early Universe give rise to classical inhomogeneities today. We believe that this aspect is no less fascinating than its other successful predictions.

We expect that models of the quantum-to-classical transition for the primordial fluctuations will continue to appear in the literature. But we are convinced that the general mechanism of this transition presented in this essay will hold true for all scenarios based on inflation.

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