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# Statement

# and

# Readings

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## Abstract

The rise of quantum mechanics and the beginnings of a quantum theory of the electromagnetic field were so closely entwined that the traditional historiography of quantum mechanics, which excludes the beginnings of quantum electrodynamics, cannot offer a complete understanding of the development of either theory. In my presentation, I will discuss the very different ideas that Jordan, Schrödinger, and Dirac pursued in their seminal works attempting to find a fully relativistic theory of the interaction of matter and the electromagnetic field. I will also look at Heisenberg's and Pauli's first realization of such a theory, and the difficulties that it encountered. In a series of papers, visitors and assistants of Pauli (Oppenheimer, Waller, Rosenfeld, Solomon, Landau, and Peierls) showed the fundamental and apparently irreparable nature of this problem. By the time of von Neumann's canonical formalization of the quantum mechanics of non-relativistic particles, a split was established between this now relatively stable theory and the program of a fully relativistic quantum field theory, which seemed to get bogged down in an increasingly complicated theoretical apparatus and invalidated by problems of infinities.

# 1 The Transformation of Field Theory

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While it was clear by 1924 that it was necessary to formulate a quantum theory of electromagnetic radiation (see Chapter **??**), it was not clear for several more years how this theory should relate to the developing quantum mechanics. It was still quite plausible to imagine this theory as a supplementary quantum mechanics for light quanta, especially since wave mechanics seemed to have given a theoretical account of wave-particle dualism. This latter conviction played a central role in Bohr's interpretation of quantum mechanics. This is quite different from the modern understanding that quantum mechanics and the quantum theory of radiation are part of an encompassing quantum field theory, which describes all types of particles and their interactions as quantum fields constructed on the basis of a classical field theory through the application of quantization procedures. These quantization procedures are generalizations of the quantization procedures established in quantum mechanics, such as canonical commutation relations or path integral formulations. Quantum mechanics then can be understood as a special case of a quantum field theory, in which the numbers of all particles are conserved. Both approaches, a quantum mechanics of light quanta and a quantized unified field theory, existed since the earliest efforts at formulating a theory of radiation based on the new framework given by quantum mechanics, even though they were not always formulated explicitly. As in the case of wave and matrix mechanics, these approaches (championed by Dirac and Jordan, respectively) led to equivalent formalisms, but without a corresponding interpretation debate. The resulting formalism developed quickly, and by 1929, a full theory of quantum electrodynamics had been formulated by Heisenberg and Pauli.

It was immediately clear, however, that this theory had severe and fundamental problems. In particular, one encountered a difficulty which in similar form had already appeared in classical electron theory: The interaction of an electron with its own electromagnetic field ("self-interaction") gave an infinite contribution to its energy. It was already clear from classical electrodynamics that this self-interaction could not be ignored as it was the cause of physical effects, such as radiation damping (the energy loss of particle due to the emission of radiation). Different diagnoses of how this problem should be addressed in the early 1930s did not lead to conclusive results. This was seen as an indication that the theory was only preliminary until the late 1940s, when techniques were developed for self-consistently removing the infinities from the empirical predictions of the theory.

## 1.1 Pascual Jordan's quantum field theory program

Pascual Jordan was the first to see quantum field theory[1] as a unified basis for all of modern physics, long before this became commonly accepted.[2] He saw his efforts not simply as directed at a quantum theory of radiation, but as part of a program to formulate a theory that represented all matter and radiation in the same way, as quantum fields. Jordan's formulation of this goal and his work towards it depended on a rather unique combination of a foundationalist universalism that would befit an Einstein or Planck, and a radical positivism that rejected vehemently the demand for a visualizable and intuitive understanding of physics.

Jordan's program grew out of his early work on radiation in the old quantum theory: In his dissertation (published as [Jordan, 1924]), he had attempted to find a way to avoid Einstein's conclusion [Einstein, 1917] that the emission of radiation by the Bohr atom had to be directed. Einstein [1925a] quickly showed that Jordan's argument rested on the physically implausible assumption that also the absorption of radiation could not be directed, i.e., that an atom could not absorb a light wave coming in from a specific direction. After this paper and a correspondence about it with Einstein, Jordan accepted Einstein's argument about the irreducibly dual nature of light. Moreover, the lessons he had learned about the statistics of the equilibrium of radiation and matter would have a decisive impact on his further development: When Jordan read Einstein's papers on the Bose statistics of the ideal gas [Einstein, 1924, 1925b], he immediately noticed the impact that the new statistics had on the theory of the interaction between radiation and matter. Jordan used the new statistics, as well as de Broglie's idea of matter waves to which Einstein had referred in order to motivate it, to study the thermodynamical equilibrium of light quanta and the ideal gas. This led him to make a strikingly novel stipulation:

> The elementary acts of dispersion [between radiation and matter] can be viewed not only as dispersion of *light radiation on material corpuscles* but also as dispersion of *matter radiation on corpuscular light quanta*; therefore, the probabilistic law will be symmetric. . . [between the densities of radiation and matter].[3]

Schrödinger had taken Einstein's theory of the ideal gas as evidence that matter and

[1] The term "quantum field theory" did not become current until c. 1960. In the time period we are considering, quantum electrodynamics was basically considered a universal theory, with no need to appeal to a more general quantum theory of fields. We use the term to distinguish Jordan's approach from the idea of a quantum mechanics of light quanta.

[2] Compared to the great amount of secondary literature on the development of quantum mechanics, there is only a small number of literature on Jordan's seminal contributions to quantum field theory, see especially [Cini, 1982] and [Darrigol, 1986]. Duncan and Janssen [2008] give a detailed account of Jordan's derivation of Einstein's fluctuation formula for radiation and the role this played in the emergence of quantum field theory.

[3] "Die Elementarakte der Zerstreuung können wir nicht nur als Zerstreuung von *Lichtstrahlen an materiellen Korpuskeln,* sondern auch als Zerstreuung von *Materiestrahlung an korpuskularen Lichtquanten* betrachten; es wird deshalb dies Wahrscheinlichkeitsgesetz symmetrisch [. . . ] sein müssen." [Jordan, 1925, 652]

radiation both had to be understood as waves [Schrödinger, 1926c]. Jordan agreed that matter and radiation were of the same nature, but he did not accept that this nature was correctly expressed by a classical wave picture. Instead, he postulated that both matter and radiation should be representable equivalently either as waves or as particles, thus establishing a complete symmetry between the two representations.

In an interview with Thomas Kuhn,[4] Jordan credited the idea of the symmetry of representations to William Duane's treatment of the scattering of light quanta by a grid [Duane, 1923]. Duane had shown that the interference on a grid, which had always been seen as a paramount wave phenomenon, could also be explained in the light quantum theory if one used the periodic structure of the grid to justify a quantized transfer of momentum to the light quanta. Jordan saw this argument as evidence that the dualism of particle and wave character of light should find its theoretical expression in a "symmetry of representations": Quantum physics should allow to represent the same physical situation equivalently in particle and in wave description. For Jordan, this symmetry of representations was a convincing argument that all previous mechanical pictures had to be insufficient. The symmetry of representations would become the fundamental heuristic principle underlying Jordan's work both in quantum mechanics and quantum field theory during the following years. Jordan claimed in the AHQP interview[5] that already at this point he was hoping that a quantum theory of waves could deliver this symmetrical representation for both matter and radiation. Although there is no direct contemporary evidence, the circumstances described above make this plausible.

In the summer of 1925, Jordan got recruited by Max Born to help in the mathematical elaboration of Werner Heisenberg's idea of *Umdeutung*. Born and Jordan [1925] showed that the matrix calculus was the appropriate mathematical form for Heisenberg's new mechanics. However, Jordan did not limit himself to the formalization of Heisenberg's ideas: the paper contains an application of matrix mechanics to the electromagnetic field. This section limited itself to a plausibility argument for Heisenberg's interpretation of the squares of matrix elements of the position vector as intensities of the emitted radiation. Nevertheless, the methods used—expressing field quantities as matrices—give an indication of Jordan's program of a quantized field theory. Also the subsequent *Dreimännerarbeit* [Born et al., 1926] contains a section on the quantization of a field, this time with a much more striking result: the derivation of Einstein's famous and puzzling fluctuation formula for radiation (See **??**) from the noncommutativity of the dynamical variables. As we know from a letter from Heisenberg to Pauli,[6] this section was written by Jordan who later considered it as "almost the most important thing I have contributed to quantum mechanics."[7]

In a study of Einstein's fluctuation formula, Paul Ehrenfest [1925] had introduced the

[4] Interview of Pascual Jordan with Thomas S. Kuhn, June 18, 1963. AHQP, Transcripts of Oral History Recordings, Microfilm 1419-03, Jordan interview 2, p. 19.

[5] Interview of Pascual Jordan with Thomas S. Kuhn, June 19, 1963. AHQP, Transcripts of Oral History Recordings, M/f 1419-03, Jordan interview 3, p. 9.

[6] Heisenberg to Pauli, October 23, 1925 [Pauli, 1979, p. 252]

[7] Jordan to van der Waerden, April 10, 1962, AHQP M/f 1419-006, p. 604. The quotes from Heisenberg and Jordan are given in [Duncan and Janssen, 2008].

model of a vibrating elastic string fixed at both ends as the simplest possible situation for the study of wave fluctuations. Each characteristic frequency of its vibration (or wave mode) can be treated as an independent harmonic oscillator. The total energy of each mode (and thus of the string as a whole) is constant. But the energy content of a small number of neighboring wave modes in a small segment of the string fluctuates because of the interference of the neighboring wave modes. Ehrenfest calculated this fluctuation and obtained only the wave fluctuation term, even if the individual wave modes were quantized in the sense of the old quantum theory. In the *Dreimännerarbeit*, Jordan quantized Ehrenfest‘s model using matrix mechanics—harmonic oscillators being one of the few things one could quantize with matrix mechanics in 1925—and discovered that the non-commutativity of the matrix calculus leads to an additional term for the energy fluctuations: it is exactly the particle fluctuation term. For the first time, Einstein's fluctuation formula had been derived from an underlying dynamical theory.

Jordan concluded his considerations with the remark:

> If one considers that the question treated here [the fluctuation of radiation] is rather removed from the problems out of which quantum mechanics arose, one will perceive the result [. . . ] as especially encouraging for the further extension of the theory. [8]

The full meaning of this remark would have eluded a contemporary reader, who could have taken it simply as an optimistic outlook on the further development of matrix mechanics. But it fits very well with Jordan's later reminiscences that he saw in this derivation the first lead to the quantized field theory he had been looking for. However, even his coauthors Heisenberg and Born were skeptical about the need to quantize the electromagnetic field [Duncan and Janssen, 2008, p. 640–642]. Jordan, on the other hand, had an even more ambitious goal: His principle of symmetries of representations implied that also matter should be represented by quantized waves in the same manner. As he claimed in [Jordan, 1927e, p. 480] and in a letter to Schrödinger, his occupation with the quantum theory of the ideal gas had suggested this further application of the theory of quantized waves. Jordan writes in the letter:

> Then your hydrogen paper [i.e., Schrödinger [1926a]] seemed to give me the hope that by following up this manner of representation also the non-ideal gas could be represented by quantized waves—that therefore a complete theory of light and matter could be developed in which it however was essential that this wave field itself already behaves in a quantum, non-classical way, [. . . ][9]

[8] "Wenn man bedenkt, daß die hier behandelte Frage doch ziemlich weit entfernt liegt von den Problemen, aus deren Untersuchung die Quantenmechanik erwachsen ist, so wird man das [. . . ] Ergebnis als besonders ermutigend für den weiteren Ausbau der Theorie betrachten." [Born et al., 1926, p. 615]

[9] "Ihre Wasserstoff-Arbeit schien mir nun die Hoffnung zu geben, daß man auch das nicht-ideale Gas in Anschluss an die obige Vorstellungsweise durch gequantelte Wellen darstellen könnte—daß man also wirklich eine vollständige Wellentheorie von Licht und Materie entwickeln könnte, wobei es jedoch wesentlich war, daß dieses Wellenfeld selber schon nicht-klassisch, quantenhaft funktionierte, [. . . ]" Jordan to Schrödinger, no date, reply to Schrödinger's letter from July 28, 1927, AHQP M/f 41 Sect. 8-009b.

Jordan saw Schrödinger's wavefunctions as a generalization of the simple one-dimensional waves that he had quantized in the *Dreimännerarbeit* and interpreted as the quantum mechanical representation of the Bose-Einstein ideal gas; he was convinced that the quantization of wavefunctions was the method necessary to apply quantum mechanics to the case of several interacting particles.[10] In the letter to Schrödinger, Jordan gives two reasons why he did not pursue this program immediately: The problem to account for Fermi-Dirac statistics, since it seemed that the wave picture would always lead to Bose-Einstein statistics, and the reservations of his colleagues Heisenberg, Pauli, and Born.

Jordan's approach was fundamentally different both from Schrödinger's and from Heisenberg's and Dirac's ideas about the application of quantum mechanics to the many-particle problem. Schrödinger was searching for a way to represent the many-body problem as the self-interaction of a continuous charge distribution. His ambition was, like Jordan's, a unified theory of matter and radiation, but he envisioned a classical field theory in which the only quantum element was the introduction of matter waves. Heisenberg and Dirac, on the other hand, had constructed symmetrical and antisymmetrical many-particle wavefunctions from single-particle wavefunctions and given phenomenological arguments why they should account for the characteristics of spectra of many-electron atoms (see chapters **??**). Dirac showed that symmetrical wavefunctions led to Bose-Einstein statistics and that antisymmetrical wavefunctions explained the Pauli exclusion principle for electrons and therefore should be the basis of a statistics for matter particles. The success of the Heisenberg-Dirac method in the explanation of atomic spectra made Jordan's much more abstract program seem superfluous. However, Jordan's approach would have immediately given a unified theory of particles and the electromagnetic field, while Heisenberg's and Dirac's approach was limited to instantaneous (and hence non-relativistic) interactions and had nothing to say on the question of a quantum theory of radiation.

The transformation theory, developed in 1926/27 by Dirac [1927a] and Jordan [1927b,c] independently, was for Jordan further evidence for his principle of symmetry of representations. It was indicative of Jordan's positivism that his transformation theory did not use the concept of a state at all; rather, what he used for the description of a physical system was the totality of all possible transition amplitudes between the values of physical quantities, the squares of which give the probability of finding the value of one quantity given the value of another quantity. Instead of specifying, e. g., one specific state of a hydrogen atom by a wavefunction, Jordan's transformation theory describes all possible states of the hydrogen atom by the transition amplitudes between a basis diagonalizing the energy matrix and a basis diagonalizing the position matrix of the electron. Jordan now identified "particle" properties with the basis diagonalizing the position matrix and "wave" properties with the basis diagonalizing the momentum matrix conjugate to the position matrix. However, behind these representations, there was no state defining the objective properties of the physical system. Since the theory is invariant with regards

[10]Interview of Pascual Jordan with Thomas S. Kuhn, June 20, 1963. AHQP, Transcripts of Oral History Recordings, M/f 1419-03, Jordan interview 4, p. 3.

to the choice of basis, the system can be described equally in particle or wave language. Therefore, neither description of the system (as a particle or as a wave) is fundamental. To Jordan, this showed that there is no preferred ontological basis in which quantum mechanics should be explicated.

This conviction about the symmetry of representations was also the background for Jordan's attack on Schrödinger's physical wave interpretation of wave mechanics .[11] Jordan agreed with Schrödinger that light and matter show analogous behavior and should be treated analogously in quantum theory. But he argued that just as classical wave optics fails for the effects that made the light quantum theory necessary, so wave mechanics alone cannot account for the particulate aspects of matter. Otherwise, there would be a disanalogy between the theories of light and matter. Schrödinger himself had to admit by the end of 1926 that his original program didn't work, although for other reasons than the ones that Jordan had given: One could construct a relativistic field theory of matter interacting with an electromagnetic field. Such a theory, based on the relativistic matter wave equation known today as the Klein-Gordon equation, had already been proposed by several physicists in 1926 [Kragh, 1984]. However, this theory gave incorrect predictions for the hydrogen spectrum (even if one neglected the relativistic corrections, which caused problems which had led Schrödinger to abandon a relativistic wave equation once before, see chapter **??**) if one included the interaction of the electron field with the electromagnetic field it produced. The continuous theory, therefore, could not be the closed theory of interacting matter and electromagnetic fields that Schrödinger was searching for [Schrödinger, 1927].

At the beginning of 1927, Jordan's program was merely a vision that had not found a concrete form. Jordan felt discouraged not only by the lack of enthusiasm among his colleagues, but also by the fact that his first tentative success with his program, the derivation of Einstein's fluctuation formula, seemed to be very specific to light quanta: It was closely tied to Bose-Einstein statistics, and offered no hints as to a generalization to matter waves, which were to obey Fermi-Dirac statistics.[12]

## 1.2 Paul Dirac's method of second quantization

The idea of a quantized field only came to the attention of a wider group of physicists[13] through Paul Dirac's "The quantum theory of the emission and absorption of

[11] [Jordan, 1927a]. See also chapter **??**.

[12] Jordan to Schrödinger, no date, reply to Schrödinger's letter from July 28, 1927, AHQP M/f 41 Sect. 8-009b

[13] Jordan thought for the rest of his life that he did not get due credit for his work: "It has always saddened me somehow that the attack on the light-quantum problem already contained in our Dreimännerarbeit was rejected by everyone for so long (I vividly remember how Frenkel, despite his very friendly disposition toward me, regarded the quantization of the electromagnetic field as a mild form of insanity) until Dirac took up the idea from which point onward he was the only one cited in this connection." ("Insbesondere hat es mir immer etwas leid getan, dass der schon in unserer Dreimännerarbeit enthaltene Angriff auf das Lichtquantenproblem so lange allgemein abgelehnt wurde (ich erinnere mich gut, wie z.B. Frenkel trotz aller freundschaftlicher Sympathie, die er mir entgegenbrachte, die Quantelung des elektromagnetischen Feldes als eine Art leichtes Irresein bei mir beurteilte), bis Dirac die Idee

radiation" [Dirac, 1927b]. Unlike Jordan's endeavors, Dirac's paper was not meant to be programmatic, but explicitly as a provisional way to treat emission and absorption of light without a fully relativistic quantum theory of matter and radiation, which Dirac saw as the central objective of his research. Paradoxically, the notion of quantizing a field appears nowhere in the paper. Dirac started from the simple one-particle Schrödinger equation and considered a time-dependent perturbation. Using the techniques developed in [Dirac, 1926a], he expressed the perturbed state $\psi$ in terms of the eigenstates $\psi_r$ of the unperturbed Hamiltonian $H_0$

$$\psi = \sum_r a_r \psi_r \tag{1.1}$$

with time-dependent coefficients $a_r$. In his own recently developed transformation theory [Dirac, 1927a], their squares $|a_r|^2$ were interpreted as the time-dependent probabilities of finding the system in the state $\psi_r$. Dirac observed that this can be interpreted as describing how a statistical ensemble of noninteracting systems (obeying classical Boltzmann statistics) reacts to an external perturbation, since the squared expansion coefficients can equally well be read as giving the ratio of systems in each eigenstate.
Perturbation theory gives for a perturbed Hamiltonian $H = H_0 + V$ the following time-dependence of the expansion coefficients:

$$i\hbar \dot{a}_r = \sum_s V_{rs} a_s \tag{1.2}$$

where the $V_{rs}$ are the matrix elements of the perturbing potential $V$ in the basis of the $\psi_r$. Dirac now showed that if one treated the $a_r$ as $q$-numbers (see chapter **??**), the same equations can be interpreted as describing an ensemble of systems obeying Bose-Einstein statistics. In this case, $N_r = a_r^\dagger a_r$ gives the number of systems in state $r$. This quantization of a wave mechanical amplitude soon came to be called second quantization. Today this term is commonly misapplied to the transformation of a classical into a quantum field theory, where it has often been pointed out that one should really talk about field quantization. In Dirac's original use of the procedure, the name is, however, entirely justified: Dirac's procedure goes from a quantum theory of a Boltzmann ensemble to the quantum theory of a Bose-Einstein ensemble, i.e., moves from one quantum theory to another. The resulting theory is nothing but a reformulation of Dirac's theory of symmetrized multi-particle wave functions for the case of mutually non-interacting particles. As such it should also be applicable to light quanta. So far, there was neither an expression for the one-particle wave-function of a light quantum nor a Hamiltonian that describes its interaction with a charged particle. Dirac's reformulation allowed one to circumvent the first difficulty and solve the second by establishing the connection between this theory of light quanta and a classical field theory: A state containing $N_r$ light quanta of energy $h\nu_r$ is identified with a light wave of intensity

$$I_r = N_r (h\nu_r) \frac{\nu_r^2}{c^2} \tag{1.3}$$

aufnahm—wonach dann künftig nur noch er in diesem Zusammenhang zitiert wurde." Jordan to Born, July 3, 1948 (AHQP M/f 1419-006, p. 596), quoted after Duncan and Janssen, 2008) where the final factor $\nu_r^2/c^2$ stems from taking into account the density of modes around the frequency $\nu_r$. By relating the operator $N_r$ to the classically measurable quantity $I_r$, Dirac had established a connection between classical and quantum theory which did not rely on the position operator, which had played that bridging role for point mechanics. The question of a spatial wave function for light quanta could thus be bracketed. As Dirac would cryptically remark a few months later, "it is always awkward to speak of the spatial coordinates of a light quantum," [Dirac, 1927d, 585] echoing similar discussions in the old quantum theory on the difficulty of localizing light quanta (see Chapter **??**). The relation between $N_r$ and $I_r$ specifically implies a relation between the classical field strength and the operator $N_r^{\frac{1}{2}}$, and therefore allows to express the interaction energy between charged particles and light quanta in terms of this operator.

As the theory was based on a many-particle wave equation for the light quanta, it implied particle number conservation. In order to be able to describe the emission and absorption of quanta, Dirac had to introduce a reservoir of infinitely many zero-momentum light quanta. With this trick, one can apply perturbation theory to obtain the transition amplitudes for the emission and absorption of radiation by an atom. Dirac's account now also included spontaneous emission, which had eluded his earlier treatment with an unquantized electromagnetic field [Dirac, 1926a]. Thus, he could connect Einstein's emission and absorption coefficients with the matrix elements of the atomic electron in matrix mechanics. He could further show the correctness of Heisenberg's assumption, which lay at the basis of matrix mechanics, that the transition probabilities between atomic states are given by the squared matrix elements of their polarization, and so gave for the first time a self-contained account of the quantum mechanics of atomic transitions.

Unlike Jordan's earlier attempt, Dirac's theory was greeted with enthusiasm. As a generally applicable description of the interaction between material systems and electromagnetic radiation in the framework of quantum mechanics, it was used in the following years to solve a variety of long-standing probelms: Dirac himself, in a follow-up paper [Dirac, 1927c], applied his radiation theory to the problem of dispersion, generalizing the account of Heisenberg and Kramers (see Chapter **??**) and finding a unified description of dispersion and resonant emission and absorption. Viktor Weisskopf and Eugene Wigner calculated the width of spectral lines [Weisskopf and Wigner, 1930]. Ivar Waller gave the first general treatment of the scattering of radiation by electrons, including the photoelectric effect and Compton scattering [Waller, 1930b]. Gregory Breit and Christian Møller treated the interaction and scattering of two electrons [Breit, 1929, Møller, 1932]. Edoardo Amaldi gave a theory of the Raman effect [Amaldi, 1929] and Maria Göppert-Mayer studied processes involving the emission or absorption of two light quanta [Göppert-Mayer, 1931].

Today, Dirac's paper is often seen as the seminal work for quantum field theory. This is somewhat ironic, as Dirac explicitly rejected the idea that his method was to be understood as the quantization of the classical field. It was important for Dirac to point out that his procedure merely established a connection between the wave mechanical (particle) formulation and an electromagnetic field theory—the two were not to be identified. He explicitly showed that the "wave function of the light quanta" is not the same as the electromagnetic field. While the former was a quantum mechanical probability amplitude, the latter was a real field, directly related to a classical energy density. The connection between the two can only be established for the case of a quantum theory describing non-interacting Bose-Einstein particles. Thus, while an ensemble of light quanta can be associated with a light wave, there is no such physical wave associated with an ensemble of matter particles such as electrons. Dirac did not see his procedure as the quantization of a classical field, in Jordan's sense, and therefore also not as an explanation of the quantum nature of radiation. He only showed the interrelation between the light quantum and field descriptions, and did not consider the latter as more fundamental.

## 1.3 From second quantization to field quantization

There was nothing in Dirac's formalism that could not be interpreted by Jordan in terms of a quantization of fields. Rather, he took Dirac's success as an indication of the viability of his own program. Jordan returned to the theory of the quantized field during his stay in Copenhagen from 19 May til 19 October 1927[14] and published in quick succession several papers on the topic. Jordan ignored Dirac's dictum that the method of second quantization should only be applicable to light quanta and not to electrons, as this assertion was in direct contradiction with Jordan's principle of symmetry of representations: In the first paper of the series [Jordan, 1927e] Jordan generalized Dirac's reformulation of many-particle wave mechanics to the case of Fermi-Dirac statistics: Jordan observed that in the case of Bose-Einstein statistics, the number operator has arbitrary integer eigenvalues, while in the case of Fermi-Dirac statistics, the number operator can only have eigenvalues 0 or 1. He now constructed an algebra of field operators that yield these eigenvalues for the number operator using Pauli's spin matrices. This construction was made possible by Jordan's concept of conjugate variables that was more general than Dirac's: While Dirac relied on commutation relations of the standard form $pq - qp = -i\hbar$, Jordan's transformation theory relied on a more general notion of conjugate variables (motivated by the need to represent angle and angular momentum as conjugate variables[15]) and allowed for a generalization of these commutation rules.

However, as Darrigol [1986, p. 232] has pointed out, Jordan's actual calculations were full of mistakes: "Although Jordan knew he was on the right track, his paper was only a sketch, full of misprints and imprecisions. The draft received by Alfred Landé resembles a bad student paper overcorrected by the professor." What had gotten lost in the imprecisions were the correct phase relations between the creation and annihilation operators. Only in the fall of 1927, Jordan would return to the topic and, with the help of Eugene Wigner, present the correct algebra (now called Jordan-Wigner second quantization) using anticommutation relations [Jordan and Wigner, 1928].

Despite its technical flaws, [Jordan, 1927e] Jordan saw the success of his generalization

[14]Copenhagen: Universitetets Institut for teoretisk Fysik. Register book for foreign guests. AHQP M/f 35 Sect. 2

[15]Interview of Pascual Jordan with Thomas S. Kuhn, June 19, 1963. AHQP, Transcripts of Oral History Recordings, M/f 1419-03, Jordan interview 3, p. 22–23.

as an indication that it would be possible to formulate a quantum field theory of matter waves and that this theory could be the basis for a unified quantum field theory for matter and radiation, in keeping with his earlier program:

> Despite the validity of the Pauli instead of Bose statistics for electrons, the results achieved so far leave hardly a doubt that a quantum-mechanical wave theory of matter can be formulated, in which electrons are represented as quantized waves in *ordinary three-dimensional* space and that the natural formulation of the *quantum theory of electrons* will have to be achieved by comprehending light and matter on equal footing as interacting waves in three-dimensional space. [...] The fundamental fact of electron theory, the *existence of discrete electrical particles*, thus manifests itself as a characteristic quantum phenomenon, namely as equivalent to the fact that matter waves only appear in *discrete quantized states.* [16]

As one can see from the letter to Schrödinger quoted on page 4, Jordan already had quite a clear idea what the corresponding matter wave theory was: Nothing else but Schrödinger's theory of classical matter waves. He was thus developing an understanding of second quantization entirely different from Dirac, as the quantization of a physical field and thus as an explanation of the corpuscular character of matter. Dirac, however, was not convinced and, in the discussions at the Solvay meeting (yet never in writing), criticized Jordan's quantization procedure as artificial and *ad hoc*. For Dirac, second quantization established a connection between a many-particle and a field theory, and one could not simply invent a different commutation algebra to cover up the fact that there was no such connection for matter. Unlike Schrödinger, however, Jordan did not attempt to find an objective physical description behind the mathematical formalism. The fact that the theory had to be formulated as a field theory in no way implied a field ontology in Schrödinger's classical sense. The antisymmetrical wavefunctions that Heisenberg and Dirac had constructed for many-particle systems were not at all physical waves but simply "a special case of the general probability amplitudes which have to be used as a mathematical tool for the description of the statistical behavior of quantized light and matter waves."[17] Transformation theory to him still implied that neither particle nor wave ontology were fundamental and therefore neither picture could be used to construct a complete description of objective reality.

[16] Die erzielten Ergebnisse lassen es kaum noch als zweifelhaft erscheinen, daß—trotz der Gültigkeit der *Pauli*schen statt der *Bose*schen Statistik für Elektronen— eine quantenmechanische Wellentheorie der Materie durchgeführt werden kann, bei der die Elektronen durch gequantelte Wellen im *gewöhnlichen dreidimensionalen* Raume dargestellt werden, und dass die naturgemäße Formulierung der *quantentheoretischen Elektronentheorie* derart zu gewinnen sein wird, daß Licht und Materie gleichzeitig als wechselwirkende Wellen im dreidimensionalen Raume aufgefaßt werden. [...] Die Grundtatsache der Elektronentheorie, die *Existenz diskreter elektrischer Teilchen*, erweist sich dabei als eine charakteristische Quantenerscheinung, nämlich als gleichbedeutend damit, daß die Materiewellen nur in *diskreten Quantenzuständen* auftreten." [Jordan, 1927e, p. 480]

[17] "[...] ein Spezialfall der allgemeinen Wahrscheinlichkeitsamplituden, welche als mathematisches Hilfsmittel zur Beschreibung des statistischen Verhaltens der gequantelten Licht- und Materieschwingungen zu benutzen sind." [Jordan, 1927e, p. 480].

In the following months, Jordan made quick progress towards a complete theory in a series of three papers with different collaborators. First, in the summer of 1927[18], he collaborated with Wolfgang Pauli, whom he visited in Hamburg during a break of his stay in Copenhagen, on a paper dealing with the quantum theory of the electromagnetic field [Jordan and Pauli, 1928]. This work made Jordan's divergent reading of Dirac even more explicit: Addressing Dirac's caveat that the wave function of the light quanta could not be interpreted as the electromagnetic field, Jordan and Pauli entirely dropped the concept of a light quantum wave function and directly quantized the classical Maxwell field theory. By making explicit the relation between the quantum operators and the classical field quantities (and not just their Fourier coefficients, as in Dirac's work), Jordan and Pauli aimed to show that the relativisitic invariance of the classical field theory carried over into its corresponding quantum field theory. To this end, they introduced the concept of a q-function, a q-number varying in spacetime, which was taken to represent the field strength in a point. The commutation relations for these q-functions can then be derived from the known commutation relations of their (q-number) Fourier coefficients and are found to be mainfestly relativstically invariant. Jordan and Pauli finally showed how the commutation relations between these q-functions could be transformed into commutation relations between operators acting on a functional of the entire field configuration.

For Jordan, the transition from q-numbers to q-functions was merely a specific application of his transformation theory to quantum field theory as a quantum theory of infinitely many degrees of freedom, as he explicated in [Jordan, 1927d], which he published in October 1927. To him his work with Pauli was merely the first realization of his field quantization program for the simple case of a free electromagnetic (radiation) field. Pauli, however saw in this formulation a fundamental innovation: the transition from standard calculus to Volterra's functional calculus, which was to be interpreted as the formal expression of the transition from quantum mechanics to quantum field theory. In the spring of 1927, Pauli had studied functional calculus[19] and wrote a short manuscript summarizing what he had learnt about its application to classical field theory.[20] Pauli's hope was that one could quantize field theory in close analogy to particle mechanics using functional calculus, once one had commutation relations for the classical field variables. This understanding was to become one of the guiding principles of his future work with Heisenberg: Instead of basing field quantization on the commutation relation of Fourier amplitudes, they based their quantization on the commutation relations for canonically conjugate field variables.

Upon Jordan's return to Copenhagen in the late summer, he wrote a second paper together with Oskar Klein [Jordan and Klein, 1927] showing that Dirac's method of second

[18]See Pauli to Bohr, 6 August 1927 [Pauli, 1979, p. 403–404].

[19]Letter from Pauli to Jordan, 12 March 1927 [Pauli, 1979, p. 386]. In the AHQP Oral history interview from 12 July 1963, Heisenberg described how Pauli had discovered functional calculus and introduced him to it.

[20]Manuscript "Zur Funktionalmathematik und der Hamilton-Jacobischen Theorie für Variationsprobleme, die aus mehrfachen Integralen entspringen" CERN Pauli Archives CERN-ARCH-POMC-007-005. We thank Karl von Meyenn and Anita Hollier (check!) for making the manuscript available to us.

quantization also worked for interacting particles. Klein had been thinking about a extension of Schrödinger's wave-mechanical electrodynamics to five dimensions, hoping that Schrödinger's problem of the incorrect self-interaction of the electron could be solved in this theory. Not only could he not solve this problem, but it also became clear to him that such a purely continuous theory could not even lead to Planck's law. When Dirac's paper on emission and absorption appeared, Klein saw that it offered a solution to the latter problem: If one quantized the wave modes one obtained Bose-Einstein instead of classical statistics (i.e., equipartition) and thus the Planck distribution.[21] He now hoped that an application of Dirac's method to matter waves, i.e. to Schrödinger's entire theory, would solve the problem of self-interaction as well. As a first step, he applied second quantization to a theory of matter waves interacting via electrostatic Coulomb forces. As he wrote to Dirac, he found that second quantization could be applied to interacting particles, i.e., that the correspondence between a representation through quantized Schrödinger waves and many-particle configuration space was not spoiled by the presence of an interaction, as long as one disregarded the self-interaction terms. These self-interaction terms, however, were infinite and even appeared in the one-particle equation, convincing Klein that Dirac's stress on the disanalogy between matter and light waves was correct after all.[22] He abandoned these attempts until Jordan's return: Jordan's success with representing Fermi statistics through anti-commutators convinced him of the merits of the quantum-field-theoretical to matter waves and Jordan recognized that the problem of self energies could be solved using the non-commutativity of the field operators.

In their joint paper, they used the concept of a q-function introduced by Jordan and Pauli to represent the charged matter field in a specific spacetime point. While in the classical theory, the instantaneous Coulomb interaction of such a field would result in an infinite self-energy, Klein and Jordan showed that this problem could be solved in the quantum theory by what is now called normal ordering of the field operators. Rather than quantizing a classical expression for the interaction energy $E$ which could straightforwardly be interpreted as the Coulomb interaction energy within the charge density $\rho$ of a charged matter wave field $\varphi(r)$:

$$E = e^2 \int\int d^3x d^3x' \frac{\rho(\vec{x})\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} = e^2 \int\int d^3x d^3x' \frac{(\varphi^*\varphi)(\vec{x})\,(\varphi^*\varphi)(\vec{x}')}{|\vec{x}-\vec{x}'|} \tag{1.4}$$

they quantized the classically trivially equivalent expression

$$E = e^2 \int\int d^3x d^3x' \frac{\varphi^*(\vec{x})\,\varphi^*(\vec{x}')\,\varphi(\vec{x})\,\varphi(\vec{x}')}{|\vec{x}-\vec{x}'|}. \tag{1.5}$$

Due to the non-commutativtity of the $q$-functions these two expressions differed in the quantized theory, and Klein and Jordan showed that the difference was exactly equal to the divergent self-energy encountered in the classical theory. This allowed for a quantum field theoretical reformulation of the (instantaneous) interaction between particles and

[21] AHQP oral history interview from 20 February 1963.

[22] Letter of Klein to Dirac from 24 March 1927. See also [Darrigol, 1986, p. 234].

demonstrated that quantum field theory can treat the many-particle problem, as Jordan had envisioned already in 1926.

By the fall of 1927, it looked like Jordan's program was on its way to be entirely successful and the capstone of the new quantum physics that had been formulated since 1925: Integrating quantum mechanics and electrodynamics into a unified theoretical framework, it promised to offer a unified understanding of matter and radiation based on the quantization of a classical theory of interacting fields. It would have resolved the paradoxes of the wave-particle dualism within a mathematically consistent model. Thus it offered a resolution to the interpretation debates between Schrödinger and the matrix mechanists that would have been more attractive to both sides than Bohr's doctrine of complementarity, since either side could see it as a confirmation of their core beliefs about the physical meaning of quantization. Heisenberg could view Schrödinger's wave mechanics as a classical theory in need of quantization, i.e., discretization:

> So I was very glad when I saw in the paper of Klein and Jordan and Wigner that actually the unrelativistic theory did allow this transformation, that you could start from the quantized waves and then go over to the Schrödinger scheme for the particles [many-particle configuration space]. I found that this was the solution of the Schrödinger problem. I mean the Schrödinger problem in the sense that Schrödinger said, "Well, why don't you talk about waves alone and forget about all the orbits of electrons and so on, and then everything will be all right." (AHQP interview with Heisenberg, 5 July 1963)

Also Born and Pauli referred to Jordan's quantum field theory program at the Solvay meeting in October of 1927 as a possible solution to the problems faced when explaining quantum effects based on a wave picture. Bohr was equally impressed and praised the work by Jordan and Klein in [Bohr, 1928]. At the same time, Schrödinger could see the theory as a confirmation of his idea of a unified theory based on a wave picture for matter; referring to the programmatic passage from [Jordan, 1927e] cited above, he wrote to Jordan in surprise:

> This is, as far as I understand, also my opinion. So far, I thought that it was decidedly rejected from Göttingen and Copenhagen. Now I am glad to see that prospects are improving that we will come together again. [23]

Nonetheless, there remained a stark difference of positions on methodological issues between Schrödinger and Jordan, possibly even more so than between Schrödinger and Heisenberg. Despite Jordan's polite answer to Schrödinger's letter, there was no indication that Jordan was changing his views already expressed in connection with transformation theory, that quantum mechanics did not allow for a reduction to classical models, be they particles or waves. A quantized field theory was for Jordan not a field theory in

[23] "Das ist nämlich, soweit ich es verstehe, auch meine Meinung. Ich dachte bisher, sie werde von Göttingen und Kopenhagen aus strikte abgelehnt. Nun freut es mich, daß die Aussichten sich mehren, daß wir wieder zusammenkommen." Letter from Erwin Schrödinger to Pascual Jordan, 28 July 1927, [Schrödinger, 2011, 427]

the sense of offering a visualizable spatio-temporal model, but an abstract mathematical formalism that had to be interpreted in a strictly positivistic sense, i.e., only through its observable predictions (See chapter ?? for Jordan's positivistic interpretation of quantum mechanics).

However, neither Jordan's positivism nor his argument for it from transformation theory (discussed in chapter ??) harmonize very well with his program for quantum field theory: Jordan's claims about the foundational character of quantum field theory imply a priority of an abstract field concept, with particles as secondary quantum phenomena. This abstract field concept, even though it does not coincide with Schrödinger's more physical concept of a matter field, retains one important characteristic of the classical field: the continuity and classical description of spacetime. No matter what representation is chosen, the states of the theory are defined on this continuum. For that reason, transformation theory does not have the same implications in quantum field theory as it does in quantum mechanics. Even though Jordan is not explicit about how he understands the application of transformation theory to quantum field theory, he seems to assume that particle and wave properties are represented by the two basic quantities of his formalism, the q-functions $\phi(r)$, describing the field strength at the position $r$, and the $b_k$, describing the amplitude of the excitation with the wavevector $k$. Although these two quantities are related by a Fourier transform

$$\phi(r) = \sum_k b_k u_k(r) \tag{1.6}$$

which resembles the Fourier transform between position and momentum eigenstates in quantum mechanics, this is a Fourier transform within the classical theory and therefore relates two different descriptions of a classical field. Hence, $\phi(r)$ cannot be identified with a particle property (i.e., a particle being in the position $r$). $\phi(r)$ only specifies the field strength at the position $r$, not a localization of the field at r. Therefore, the Fourier transform is not the formal expression of a symmetry between wave and particle representations, unlike in the case of quantum mechanics. Thus, Jordan's quantum field theory is not symmetrical between wave and particle representations and so does not confirm positivism in the same way that he believed transformation theory did. Rather, one could say that wave and particle picture are represented by Jordan's field theory the Dirac-Heisenberg many-particle theory of symmetrized or antisymmetrized wave functions. But since such a many-particle theory for light quanta still did not exist and had only been elegantly bypassed by Dirac, a quantum theory of electrodynamics could only be envisioned as the quantization of a field theory and not equivalently as a many-particle quantum mechanics.

Also methodically, Jordan's grand foundationalist visions are at odds with his positivism: According to the 19th century understanding of positivism, physical theory should describe, not explain. But Jordan himself kept invoking the explanatory power of quantum field theory as a justification of its fundamental nature, e.g., in the above quote from [Jordan, 1927e, p. 480]. Despite these tensions, Jordan could maintain his positivism by emphasizing the differences between his quantized fields and classical fields. He frequently stressed that the quantum field did not offer hope for picturability in the classical sense. Therefore Jordan could maintain that, although quantum field theory offers a unified foundation of physics, it does not offer a visualizable physical model of the world. All it provides are probability amplitudes connecting possible observations, like in the case of transformation theory. However, this is a much weaker argument than in the case of the explicit argument for the possibility of different representations—it does not exclude the possibility that a *non-classical* but still spatiotemporal field picture could eventually be found as a consistent model for quantum field theory.

## 1.4 Relativistic Theory of the Electron

Both Jordan and Dirac acknowledged that their proposals so far had not yet delivered a theory of interacting matter and radiation that was relativistically invariant. The only theory which fulfilled this demand was the classical theory based on the Klein-Gordon equation, discussed at the end of section 1.1. After the success of Klein and Jordan, one could have expected that the self-interaction difficulties which Schrödinger had encountered, would be removed if one quantized this classical theory. However, the description of matter through Klein Gordon waves had further difficulties: It did not describe the spin of the electron - in fact, a relativistic account of electron spin was entirely lacking. Such an account, which could then form the basis for the full quantum theory of electrodynamics, was provided by Dirac in 1928. He objected to the Klein-Gordon equation for reasons entirely different from those, who viewed it as a classical field theory that was to be quantized. He tried to understand the Klein-Gordon equation as the central equation of a relativistic quantum mechanics, and found it to be unsuited for this role.
The problem of a relativistic generalization of quantum mechanics had been foremost on Dirac's mind from early on: When considering the scattering of radiation on matter in matrix mechanics [Dirac, 1926b], he was not interested, like Jordan, in what it said about the nature of matter and radiation, but rather in the fact that an account of Compton's formulas required a relativistic treatment that matrix mechanics so far did not give. At the time he hoped that a symmetrical treatment of time and space variables as q-numbers would provide the desired relativistic extension. Dirac had to abandon this idea when he formulated his transformation theory in [Dirac, 1927a]. In this formulation of quantum mechanics, it was no longer only time-independent properties of the system, such as transition frequencies and intensities, that were being measured. It introduced the notion of measuring time-dependent dynamical variables, the time being interpreted as the instant at which at which a measurement of this dynamical variable is performed. This introduced an asymmetry between time and position that could not be removed by a relativistic formulation of the algebraic relations between their q numbers.

In [Dirac, 1928a], the paper in which Dirac presented his relativistic wave equation for the electron, it was his explicit goal to reconcile his transformation theory with relativity. The Klein Gordon equation

$$\left(\frac{\partial}{\partial x_\mu} - ieA^\mu\right)\left(\frac{\partial}{\partial x^\mu} - ieA_\mu\right)\psi = \frac{m^2c^2}{\hbar^2}\psi \tag{1.7}$$

where $A_\mu$ is the electromagnetic four-potential, could not form the starting point for this reconciliation. Walter Gordon had introduced an oscillating charge density

$$\rho_{mn} = -\frac{e}{2mc^2}\left[i\hbar\left(\psi_m\frac{\partial\psi_n^*}{\partial t} - \psi_n^*\frac{\partial\psi_m}{\partial t}\right) + 2eA_0\psi_m\psi_n^*\right] \tag{1.8}$$

which served as a classical source for the radiation emitted in transitions between the stationary states $\psi_m$ and $\psi_n$. Dirac made a somewhat cryptic remark that this charge density was suitable to calculate measurement probabilities for the particle's position, but not for any other observables, such as momentum. This seems to be a more general statement of a problem that Johann Kudar, who was working with Pauli in Hamburg at the time, describes in a letter to Dirac:[24] For the non-relativistic theory, one could translate between the wave and matrix mechanical descriptions, by taking the wave-mechanical density $\rho_{mn}$ (regardless of whether one interpreted it as a probability or a classical charge density) and constructing, e.g., the Heisenberg $x$ matrix through the relation $x_{mn} = 1/e\int x\rho_{mn}dx$. If one uses Gordon's density to construct $x$ in this manner, the analogously constructed matrix $x^2$ is not the square of the matrix $x$. Thus, there was no matrix mechanics corresponding to the Klein-Gordon wave mechanics, and this meant that the theory did not fit into the general probabilistic framework of his transformation theory.

Dirac argued that the reason why the Klein-Gordon equation did not conform to transformation theory was that it was second order in the time derivatives and thus not of the form of the generalized Schrödinger equation in transformation theory:

$$H(q, -i\hbar\frac{\partial}{\partial q})(q'|\alpha') = i\hbar\frac{\partial}{\partial t}(q'|\alpha') \tag{1.9}$$

where $(q'|\alpha')$ is Dirac's expression of Schrödinger's wave function as a transformation function between the bases $q'$ for position and $\alpha'$ for energy eigenstates (in more standard notation this would be $\psi_\alpha(q)$).

In a talk he gave in Leipzig later in the year [Dirac, 1928b], Dirac got more explicit about the physical meaning of this problem. The integral of Gordon's electric density over a arbitrary region of space is represented by an operator that does not necessarily have eigenvalues 0 and $-e$. Therefore, it contradicts our expectation that when measuring whether an electron is within that region, we will always get a yes or no answer. Only a density of the form $\rho_{mn} = \psi_m^*\psi_n$ would give the correct eigenvalue spectrum. In particular, it could not depend on the time derivative of the wave function, as was

[24] J. Kudar to P. Dirac, 21 December 1926, AHQP 59-002. Pauli had already expressed similar reservations about the Klein-Gordon equation in a letter to Gregor Wentzel from 5 July 1926, in which he pointed out that solutions of the Klein-Gordon equation corresponding to different energy values will not be orthogonal in regards to the standard scalar product. This orthogonality, however, is essential for the standard construction of matrices from the solutions of the wave equation, i.e. for the translatability between matrix and wave mechanics, which corresponds to Dirac's complaint above. Pauli still was willing to entertain the possibility that the translation scheme could be modified (Letter to Schrödinger from 12 December 1926). In Dirac's general transformation theory, however, Pauli's translation scheme had become a special case of a basis transformation in a generalized state space and thus could not be modified without affecting the fundamental features of quantum mechanics.

the case for Gordon's density. But even if this expression was chosen for the density, it would run into a further contradiction: Since the Klein Gordon equation was of second order in the time derivatives, such a density could be made to vary arbitrarily by specifying arbitrary initial values for the time derivatives and therefore could not obey a conservation principle. Dirac thus obtained a further, physical motivation for his demand that the correct relativistic wave equation be linear in the time derivatives. This then, was his starting point for the derivation of his new wave equation. Dirac's famous derivation is still reproduced in most textbooks of relativistic quantum mechanics and is discussed in [Kragh, 1981]. While the details of this derivation are not important to our story, the decisive point for the development of quantum field theory is that Dirac managed to construct a relativistic analogue of the Schrödinger equation which gave a satisfactory account of the full interaction with the electromagnetic field, even though Dirac did not solve the problem of the existence of states of negative energy, and his proposal to deal with it through hole theory [Dirac, 1930] remained controversial for the coming years. Nevertheless, Dirac's theory did not yet incorporate the quantization of the electromagnetic field, which Dirac himself had shown in 1927 to be necessary for the correct treatment of the emission and absorption of radiation by matter. Therefore, it could not be a full quantum theory of the interaction of radiation and matter. On the other hand, it was clear that any such theory would have to incorporate Dirac's treatment of the relativistic quantum states of the electron.

## 1.5 The formulation of quantum electrodynamics

Heisenberg and Pauli started their cooperation on a full quantum theory of electrodynamics[25] in late 1927,[26] even before the publication of Dirac's relativistic electron theory. Inspired by the work of Jordan and Klein, which showed the possibility of representing interactions in a quantum field theory, they originally attempted to construct a relativistic quantum field theory based on a five-dimensional unified field theory that Klein had proposed shortly before Klein [1928] with the explicit purpose of having it serve as the basis for a quantum field theory.[27] Such a quantum field theory would be a generalization and unification of Jordan's work on matter field quantization and Pauli's and Jordan's work on the relativistically invariant quantization of the free electromagnetic field. It was to be formulated using the functional calculus that Pauli put great hopes in since his paper with Jordan. The essential dynamical equation of the theory then was a functional generalization of the Schrödinger equation:

[25] The title of Schweber's "QED and the Men Who Made It" [Schweber, 1994] reflects a common perception that quantum electrodynamics did not exist before the late 1940s and the work of Tomonaga, Schwinger, Feynman and Dyson. This idea is, however, not even supported by the text of the book itself, as Schweber consistently (and correctly reflecting historical usage) refers to the theories of the late 1920s as "quantum electrodynamics."

[26] Surprisingly nothing is preserved of their apparently so extensive correspondence of these first months of working on quantum electrodynamics (Pauli to Kronig, 22 November 1927 [Pauli, 1979, 415–416]; Heisenberg to Jordan, 7 December 1927, AHQP 14-002-017).

[27] As reported by Klein in the interview by T. S. Kuhn and J. L. Heilbron, session VI, July 16, 1963.

$$E\Psi\left[Q(x)\right] = H\left[Q(x), P(x)\right]\Psi\left[Q(x)\right] \tag{1.10}$$

where $\Psi$ is a functional on the space of field configurations $Q(x)$ and $H$ is a functional operator. $H$ could be obtained by finding an expression for the total field energy as a functional of field strengths $Q(x)$ and canonically conjugate field momenta $P(x)$ and replacing them by functional operators consistent with the commutation relations. Their first attempts seem to have focused on finding such an expression $H$ for the total energy of interacting radiation and matter fields. The only such expression they could construct, however, included not only the infinite electromagnetic zero-point energy, but also an infinite electromagnetic energy caused by the interaction with the matter field, the electron self-energy. This energy could not be removed by modifying and reordering the interaction term in a generalization of the Klein-Jordan procedure. This difficulty is described by Pauli in several letters (Pauli to Kramers, February 7 1928; Pauli to Dirac, February 17 1928). The reason for the failure of the Klein-Jordan procedure was summarized in a footnote of the final paper: Klein and Jordan had used the non-commutativity of field operators to eliminate the self energy. However, if the electromagnetic interaction was not expressed as the self-interaction of a matter field, and thereby as a product of non-commuting operators, but rather as an interaction between an electromagnetic potential and a matter field, whose respective operators commuted, reordering made as little difference in the quantum theory as in its classical counterpart.[28] They decided to ignore the problem for the time being, and to use a Hamiltonian as basis for quantization for which the self-energy diverged. Since in the meantime, Dirac's relativistic electron theory had been published, they shifted their focus from Klein's five-dimensional theory, to a classical field theory of a Dirac electron wave interacting with an electromagnetic field. A Hamiltonian for such a theory was first proposed by Hugo Tetrode, in the context of giving a wave-mechanical interpretation of Dirac's electron theory, as a first step towards a general relativistic extension of Dirac's special relativistic theory [Tetrode, 1928].
But another problem appeared, which seemed to make the entire quantization process impossible. A Hamiltonian theory of the interaction between matter and electromagnetic fields had to be formulated in terms of potentials, not in terms of fields. Already in his work with Jordan, Pauli had commented that they had not been able to construct relativistically invariant commutators for the potentials. A solution appeared to be the use of equal time commutators, i.e., defining the commutation relations only between field operators at different points in space at a fixed time, a procedure which was not manifestly relativistic. This formulation had the advantage that the commutators were of the same form as the canonical commutation relations of non-relativistic quantum mechanics[29]. A field theory could thus be quantized in full analogy to the canonical quantization of classical mechanics, if one defined the field momentum $P(x,t)$ canonically conjugate to the field strength $Q(x,t)$ through

[28] In the letter to Dirac, Pauli states his conviction that the problem can thus only really be solved by a theory unifying electromagnetic and matter waves.

[29] This is mentioned in a letter from Pauli to Weyl, January 29 1928

$$P(x,t) = \frac{\partial L\left[Q, \frac{\partial Q}{\partial t}, \frac{\partial Q}{\partial x_i}\right]}{\partial \frac{\partial Q}{\partial t}} \tag{1.11}$$

where $L$ is the Lagrangian density of the field theory being quantized. In electrodynamics, however, this led to a new complication: The canonically conjugate momentum of the electrical potential vanished identically and could thus not obey the necessary canonical commutation relation. In addition, the equation of motion corresponding to this component, was, due to to the vanishing of the momentum, a non-dynamical constraint (the Maxwell equation div$\vec{E} = \rho$) for the field operators at a given time. And this constraint was incompatible with those canonical commutation relations that *could* be implemented. If one included matter fields, this constraint even implied that electromagnetic and matter wave operators at different space points did not commute. Hence, canonical commutation relations analogous to those of quantum mechanics could not be defined for electrodynamics.

In January 1929, Heisenberg came up with a way to circumvent this difficulty[30]. One added to the Lagrangian an auxiliary term containing the time derivative of the electric potential, proportional to a parameter $\epsilon$, which was to be set to zero in all physical observables calculated from the theory, thus returning to a theory that corresponded to classical electrodynamics. In this way, the scalar component of the electromagnetic potential was artificially made dynamical, had a non-vanishing canonical momentum and could thus be canonically quantized. Pauli visited Heisenberg in Leipzig on January 19-20, where they decided to implement this method[31] and, upon Pauli's return to Zurich, immediately started working on the paper in somewhat of a rush, since Heisenberg was bound to leave for the United States in early March[32]. Pauli was still dissatisfied[33] with several elements of the paper they finally submitted on March 19 Heisenberg and Pauli [1929a]: for one, with the introduction of further auxiliary terms, necessary to ensure that there were periodic solutions to the wave equations for all four components of the electromagnetic potential. Periodic solutions were necessary, because Pauli's functional approach did not carry as far as he had hoped (another major cause for Pauli's dissatisfaction): For all practical calculations, one had to Fourier expand the field strengths and revert to a formulation with creation and annihilation operators. But his main worry concerned the old problem of the electron's self energy. In the paper, they calculated the perturbation of the energy, ignoring terms of order $(v/c)^2$. The result was the well-known divergent Coulomb self-energy, which could be ignored with the argument that it was a constant independent of the state of the electron. However, this was not clear for the terms containing $(v/c)^2$ and Pauli was afraid that these terms might "completely ruin the theoretical results."[34]

This certainly was the case in classical electron theory: A classical point-like electron

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[30] Pauli to Bohr, January 16, 1929

[31] Heisenberg to Jordan, January 22, 1929, AHQP, MF 18

[32] Pauli to Klein, February 18, 1929

[33] He expressed this dissatisfaction in a letter to Klein, on 16 March 1929.

[34] "[...] die theoretischen Ergebnisse vollständig ruinieren werden." ibid.

causes an infinitely strong field, which in turn has an infinite total energy. While this energy can be subtracted away in the static (Coulomb) case, it causes additional problems for the case of accelerated charges. The infinitely strong field then gives the electron an infinite inertia, which can be interpreted as an infinite back-reaction of the radiation emitted by the electron. This fact makes it impossible to give a consistent account of the motion of point charges in classical electrodynamics: One can only calculate the motion of a point particle in an undynamical field, or the radiation emitted by a point particle with a given trajectory, but cannot combine both types of calculations into a consistent, closed dynamical framework.

Pauli explored this question together with his new assistant Oppenheimer. Within a few months they had confirmed Pauli's worries: When taking into account higher order terms beyond the Coulomb self-interaction, the self-energy was no longer a constant and hence the observable differences between energy levels also turned out infinite.[35] Heisenberg, still in the US, was less worried about these problems and instead pursued the question of how to quantize the electromagnetic potential in a more elegant manner without auxiliary terms.[36] A straightforward way out of the problem of the vanishing momentum, was to simply fix the value of the scalar potential and make it a non-dynamical variable, which trivially commuted with all other field variables. The problem with this solution was that then also the equation of motion for the scalar potential no longer resulted from the variation of the Lagrangian and therefore the dynamical problem was underdetermined. Heisenberg now realized that this underdetermination could be circumvented by exploiting a symmetry that Weyl had found and presented one year earlier in his book "Gruppentheorie und Quantenmechanik".

As already Schrödinger had realized [Schrödinger, 1926b, 133], the most natural introduction of electromagnetic coupling into relativistic wave mechanics, was to replace the four-momentum operator $\left(-i\hbar\frac{\partial}{\partial x_\alpha}\right)$ with $\left(-i\hbar\frac{\partial}{\partial x_\alpha}+\frac{e}{c}\phi_\alpha\right)$. Weyl realized that this form of coupling implied that the wave equation was invariant under a substitution:

$$\begin{aligned}\psi(x_\beta) &\rightarrow e^{-\frac{ie}{\hbar c}\lambda(x_\beta)}\psi(x_\beta)\\ \phi_\alpha(x_\beta) &\rightarrow \phi_\alpha(x_\beta)+\frac{\partial\lambda(x_\beta)}{\partial x_\alpha}\end{aligned} \tag{1.12}$$

where $\lambda(x_\beta)$ is an arbitrary space-time function. Since Weyl had called a similar invariance in his unified field theory of 1918 "gauge invariance" (*Eichinvarianz*), he continued to use this name for it. Heisenberg realized that if one fixed the value of the scalar potential, a residual gauge symmetry remained, in which $\lambda(x_\beta)$ did not depend on time. To this symmetry Heisenberg now applied the principle that to each symmetry of the Hamiltonian corresponds an operator that commutes with the Hamiltonian. As Dirac had observed, such an operator in turn corresponded to a physical quantity conserved in time [Dirac, 1929]. This implies that any solution of the dynamical problem can be clas-

[35] Pauli to Bohr, July 17 1929

[36] Heisenberg to Pauli, July 20 1929; also Heisenberg to Bohr.

sified according to the constant value that this physical quantity assumes. The conserved quantity corresponding to the residual gauge symmetry turned out to be

$$C = \mathrm{div}\vec{E} - \rho. \tag{1.13}$$

One could thus first solve the dynamical problem without the first Maxwell equation and then pick those solutions for which $C = 0$, i.e., for which the first Maxwell equation was fulfilled. There was thus a subsidiary condition on the quantum state $\psi$, which had to fulfill the equation $C\psi = 0$. This had the additional advantage that the first Maxwell equation was no longer an operator identity and thus did not interfere with the canonical commutation relations. Although this new quantization method was not manifestly Lorentz invariant, Heisenberg and Pauli convinced themselves "that all statements about gauge invariant quantities [...] fulfill the demand of relativistic invariance."[37] Heisenberg and Pauli presented this approach in a second paper, submitted while Heisenberg was still in America [Heisenberg and Pauli, 1929b].

In spring of 1929, Enrico Fermi made an independent attempt at quantizing electrodynamics, based on Dirac's radiation theory Fermi [1929]. He quantized all four components of the electromagnetic potential as harmonic oscillators, but did not express the electrons as second-quantized fields. Unlike Heisenberg and Pauli, who had to introduce additional degrees of freedom and auxiliary terms to make all components of the electromagnetic potential oscillate harmonically, Fermi simply worked in the Lorenz gauge $\partial_\alpha \phi_\alpha = 0$, in which all four components fulfill the wave equation already in the classical theory. Originally, he did not worry about the fact that these components were not all dynamically independent, but rather coupled precisely through the gauge condition. Heisenberg and Pauli, who became aware of Fermi's work, before completing their second paper, realized that this Lorenz gauge condition (and its time derivative) could be taken as subsidiary conditions on the quantum state, with the same justification as they had used in their second paper. Fermi adopted this interpretation and presented it in his influential Michigan Summer School lecture in the following year Fermi [1932]. His simpler approach, which did not use functional methods or matter wave quantization became the standard formulation of quantum electrodynamics for the next two decades. Whichever method one preferred, by the end of 1929 it was clear that the quantization methods of Jordan and Dirac allowed for the formulation of a quantum theory corresponding to classical electrodynamics.

## 1.6 Problems of the quantum field theory program

Two major questions remained for Heisenberg and Pauli's theory: Did this new quantum electrodynamics go beyond Dirac's quantized theory of radiation? And what did the infinities that appeared in the calculations mean for the applicability of the theory?
While initially Heisenberg and Pauli were looking for new applications of their theory,

[37] "[...] daß alle Aussagen über eichinvariante Größen [...] der relativistischen Invarianzforderung genügen" [Heisenberg and Pauli, 1929b, 178].

within the next few months this search became moot, as the first question was answered negatively. That second quantization of the electron field was not necessary to formulate a quantum electrodynamics, was implicitly acknowledged by Heisenberg and Pauli, who, in their second paper, transformed their quantized field theory of the electron into a description in a many-particle configuration space. QED could thus be viewed, in both formulations, as the interaction of a set of quantum mechanical particles interacting with the quantized electromagnetic field. This quantized electromagnetic field contained, in addition to the radiation components, either an unperiodic longitudinal component (Heisenberg-Pauli) or periodic longitudinal and scalar components (Fermi). These additional components, however, could be exactly (i.e., nonperturbatively) eliminated from the theory using the auxiliary conditions on the wavefunction. This was shown for the Heisenberg-Pauli formulation by Oppenheimer, in a paper which he completed shortly after his return to the USA from Zurich [Oppenheimer, 1930], and by Fermi for his own formulation for the first time at the Michigan Summer School. What remained after this elimination was simply the radiation field and an instantaneous Coulomb interaction between the electrons, including the infinite Coulomb self-interaction.[38] Pauli himself observed in a letter to Oscar Klein from 10 February 1930 that "the relationship of our theory to Dirac's radiation theory is indeed very close":[39] The full quantum electrodynamics added only an instantaneous Coulomb interaction (and self-interaction) to Dirac's radiation theory. The two theories were otherwise equivalent.

Addressing the second question, Oppenheimer published the details of the calculations he had performed with Pauli in the summer of 1929 in Oppenheimer [1930]. He explicitly showed that the perturbation terms derived from quantum electrodynamics led, not only to infinite energies for a given state (as had already been observed in the first paper of Heisenberg and Pauli), but also to infinite energy differences between states, i.e., to an infinite displacement of spectral lines. The self-interaction of the electron, which Oppenheimer found responsible for these divergences, could only be ignored in the nonrelativistic limit (where the only the harmless Coulomb self-interaction remained) and possibly in the case of the calculation of line splittings, where the infinite line displacements might cancel. Aside from this special case, he concluded that the theory could give no account of "any problem where relativistic effects are important."

After Oppenheimer had left Zurich, Pauli, while publishing nothing on the subject himself, continued to prod his students and visitors to search for manifestations and the roots of the problem of infinities in QED. Lew Landau and Rudolf Peierls attempted a reformulation of QED in which also the photons would be described by wave functions in many-body configuration space (of variable dimension), remarking that such formal manipulations did not remove the infinities [Landau and Peierls, 1930]. Even though they

[38] In his 1936 textbook [Heitler, 1936], Walter Heitler showed that already for classical electrodynamics, the interaction of particles and the field can be split up into an instantaneous Coulomb interaction and an interaction with plane electromagnetic waves. In the second edition [Heitler, 1944], Heitler showed that this corresponds to a gauge transformation to what is today known as the Coulomb gauge.

[39] "Die Beziehungen unserer Theorie zur Diracschen Strahlungstheorie sind in der Tat sehr enge." [Pauli, 1985, p.2-3]

could show that the resulting Schrödinger equation was equivalent to Heisenberg-Pauli QED, it turned out that this Schrödinger equation allowed negative energy solutions, just like the Dirac equation.[40] This implied that the coupling to the electron would induce transitions to states with negative energy light quanta, rendering the Schrödinger equation phyiscally meaningless. Ivar Waller, visiting from Upsala, extended Oppenheimer's calculation to the case of a free electron with a continuous energy spectrum (Oppenheimer had assumed a discrete energy spectrum for the electron) [Waller, 1930a]. Just as Oppenheimer, Waller found that also the energy differences between states of different electron momentum became infinite.

Leon Rosenfeld, who was Pauli's assistant in Zurich in 1929/30, investigated the gravitational field energy induced by a light quantum in the absence of matter[Rosenfeld, 1930]. This calculation seemed to throw a new light on the question of self-energy, since a classical electromagnetic wave would not lead to a singularity in the gravitational field. The infinity he encountered in the quantum theoretical result was therefore thought to indicate that there were divergences intrinsic to the quantum theory which were not the consequence of a classical singularity.[41] One could, however, expect that this infinity was caused by the zero-point energy of the electromagnetic field.

Heisenberg even believed that he could pinpoint the origin of the divergent self-energy of the electron in the electromagnetic zero-point energy[Heisenberg, 1930]: He argued that the electron self-energy could be made to vanish, if one were able to construct a vacuum electromagnetic field configuration with vanishing energy and momentum. Such a configuration was, however, impossible in a quantum theory precisely due to the presence of the zero-point energy, which was thus, in effect, responsible for the infinite self-energy. Heisenberg acknowledged that the zero point energy was absent in the configuration space formulation of Landau and Peierls, but casually dismissed the "formal tricks" that Landau and Peierls had used in their work, which included the use of integral operators in the wave equation. Jacques Solomon and Rosenfeld reinterpreted the wave function for photons that Landau and Peierls had introduced as a (complex) representation of the classical electromagnetic field and showed that the corresponding quantum field theory did not contain a zero-point energy [Rosenfeld and Solomon, 1931]. However, Solomon published another paper shortly later, during a stay in Zurich, in which he showed that this reformulation did not even remove entirely the infinite interaction energy with the gravitational field that Rosenfeld had derived. He concluded that "The central problem of quantum theory does not consist in the removal of this infinite additive [zero-point] energy [. . . ], but in a correct formulation of the interaction."[42]

In 1931, Rosenfeld, now in Copenhagen, calculated the perturbation of the energy for

[40] Landau to Peierls, 6 February 1930 [Peierls, 2007, 75-76].

[41] However, the claim that this case was fundamentally different ignored that Heisenberg and Pauli's QED was not a quantization of a Lorentz-type electron theory. The classical field theory which corresponded to Heisenberg and Pauli's QED, the Schrödinger-Tetrode theory of matter waves interacting with the classical electromagnetic field, included no singularities either.

[42] "Das Hauptproblem der Quantentheorie besteht also nicht in der Beseitigung dieser unendlichen additiven Energie [. . . ], sondern in einer korrekten Formulierung der Wechselwirkung. [Solomon, 1931, 170]

the special case of an electron in a harmonic oscillator potential [Rosenfeld, 1931]. Since here all the unperturbed wave functions are easily normalized and well-behaved (as opposed to the plane waves of a free electron), the fact that the perturbation is still infinite underlined the fundamental nature of the problem. In this work, Rosenfeld also rebutted a suggestion by Dirac, that the infinities might be removed by making a non-relativistic approximation in which the effects of retardation were consistently neglected. After all, Dirac had never intended for his radiation theory to be a fully relativistic theory of quantum electrodynamics, so he still had hope that a consistent non-relativistic limit should be possible. However, Rosenfeld could show that the infinities only got worse in such an approximation. Thus it was not the ambitions of Heisenberg and Pauli to formulate a fully relativistic QED that led to the problem of infinities, but the quantization of the infinite number of radiation modes that Dirac had performed in 1927. Triumphantly, Pauli demanded of Dirac that he "sufficiently stress in all future publications the flawedness in principle of the foundations of your theory of photons." [43]

Von Neumann's book "Mathematical foundations of quantum mechanics" [Neumann, 1932] marked the final codification of nonrelativistic quantum mechanics (see chapter **??**). In this book, the theory of radiation appeared only as a fragment, unconnected with the axiomatic approach of the rest of the book. Von Neumann limited himself to a recounting of Dirac's original derivation of the emission and absorption coefficients. While quantum mechanics by 1932 had reached a canonical form, the debates about its interpretation notwithstanding, quantum field theory remained unfinished and provisional. The hopes for a unified quantum theory of matter and radiation gave way to a split between an established quantum mechanics and a highly problematic quantum field theory. Only through this split, our modern distinction between the two fields was created.

The quandaries of quantum field theory were not to find an end anytime soon: While various proposals to remove the infinities were made in the following years, it remained unclear how a general theory without inconsistencies could be built up. Possibly even more damaging to the provisional quantum electrodynamics that had been established was the fact that it did not offer measurable predictions going beyond ones that could be guessed from a use of the correspondence principle Heisenberg [1931], and there was no way to extract these predictions from the theory other than comparing them to the results gained from correspondence arguments [Rosenfeld, 1932]. One possible diagnosis was that the problem of infinities was inherited from classical electrodynamics. One could attempt to modify the classical field theory, as for example Born and Infeld did, turning to a program similar to that of the unified field theory program as a prerequisite for a quantized field theory. Alternatively, one could see the problem as indicating the limitations of the procedure of quantizing a classical theory. In this spirit, Jordan [1929] proposed that a new autonomous quantum field theory would have to be found to solve the problem. Even more radically, Landau and Peierls [1931] proposed that the uncertainty relations made field strengths unmeasurable in principle and therefore undermined

[43] "[. . . ] die prinzipielle Fehlerhaftigkeit der Grundlagen Ihrer Photonentheorie in Ihren künftigen Publikationen genügend hervorheben werden [. . . ]" Pauli to Dirac, April 21, 1931, [Pauli, 1985, p.76]

the concept of a quantum field. Unlike for the case of observables in quantum mechanics, no operational definition could be given to the notion of a field at a point in space and time. Although the specific arguments of Landau and Peierls could be countered by Bohr and Rosenfeld [1933] through a detailed consideration of measurement of field quantities in arbitrarily small regions of spacetime, a general distrust towards the very foundations of quantum field theory remained through the following two decades. Still, to a certain degree, quantum field theory remained an active field of research through the 1930s and 40s: Phenomena such as the creation and decay of cosmic ray particles and nuclear beta decay could only be described in this framework, and quantum electrodynamics had shown how to gain a modicum of empirical predictions from a quantum field theory, in spite of its inconsistencies.

Only after World War II did the observation of the Lamb shift offer a first empirical confirmation of a quantum electrodynamical effect that went beyond the predictions of correspondence arguments. This led to a resurgence in interest in the quantum field theory program—not as a foundational endeavor of unification, but as a phenomenological theory for calculating radiative corrections. The most important technical advance was the introduction of Feynman diagrams which allowed for vastly simplified calculations in perturbation theory.[44] New renormalization techniques were developed for absorbing the infinities into unobservable corrections to the masses and charges of the elementary particles. With these developments, the expectation that quantum field theory would soon give way to a more general foundational theory was abandoned for the time being: The difficulty of the infinities was relegated to a high energy theory which had no effects for the phenomena observable in contemporary experiments. Against all expectations, however, this provisional theoretical structure was able to accommodate all the theoretical advances of the following decades and was developed into a theory encompassing the whole spectrum of newly discovered elementary particles and all their interactions, with the conspicuous exception of gravity. While the quantum field theory program therefore was successful beyond anyone's expectations, its fundamental problems, and thus the problem of a unified quantum physics, remain unresolved to this day.

[44] See [Schweber, 1994] for a treatment of the history of quantum electrodynamics after World War II

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## Studien zur Entwicklung von Mathematik und Physik in ihren Wechselwirkungen

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# Mathematics meets physics

A contribution to their interaction in the 19th and the first half of the 20th century

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# Mathematical Foundations and physical Visions: Pascual Jordan and the Field Theory Program

**Christoph Lehner**

# 1 Introduction

The work of Pascual Jordan (1902 – 1980) offers rich material for a study of the complex interactions between mathematics and physics in the twentieth century, and especially for its possibly most eventful period, the years 1925 – 1927 when modern quantum mechanics and quantum field theory were established. Jordan was truly a scion of the unique closeness if not amalgamation of physics and mathematics characteristic for Göttingen in the days of Felix Klein and David Hilbert. Within two years of his arrival there in 1922, he had been a student assistant with the theoretical physicist Max Born revising his article «Dynamik der Kristallgitter» [Born, 1923], with the mathematicians Richard Courant and David Hilbert working on the textbook *Methoden der Mathematischen Physik* [Courant and Hilbert, 1924], and with the experimentalist James Franck coauthoring the review article «Anregungen von Quantensprüngen durch Stöße» [Franck and Jordan, 1926]. The present contribution will discuss the connection of this educational background with Jordan's program and achievements in quantum field theory.

Jordan was the earliest and most ambitious visionary of the quantum field theory program: long before this became commonly accepted in the second half of the twentieth century, he saw in quantum field theory a unified basis for all of modern physics.[1] Jordan's formulation of this goal and his work towards it depended on a rather unique combination of a foundationalist universalism that would befit an Einstein or Planck, and a radical positivism that rejected vehemently the demand for a visualizable and intuitive understanding of physics. While it is not hard to discern these two tendencies in Jordan's work and see the tension between them, it is less obvious to understand how they relate to the balance between mathematics and physics in Jordan's work. Nevertheless, I will claim that there is an intimate connection between the two relationships.

[1] Jordan's seminal contributions to quantum field theory are described in more detail in [Cini, 1982] and [Darrigol, 1986]. Duncan and Janssen [2008] give a detailed account of Jordan's derivation of Einstein's fluctuation formula for radiation and the role this played in the emergence of quantum field theory.



# 2 Neither waves nor particles

In his dissertation,[2] Jordan had attempted to find a way to avoid Einstein's conclusion [Einstein, 1917] that the emission of radiation by the Bohr atom had to be directed. Einstein [1925a] quickly showed that Jordan's argument rested on the physically implausible assumption that also the absorption of radiation could not be directed, i. e., that an atom could not absorb a light wave coming in from a specific direction. After this paper and a correspondence about it with Einstein, Jordan accepted Einstein's argument about the irreducibly dual nature of light. However, the lessons he had learned about the statistics of the equilibrium of radiation and matter would have a decisive impact on his further development: When Jordan read Einstein's papers on the Bose statistics of the ideal gas [Einstein, 1924, 1925b], he immediately noticed the impact that the new statistics had on the theory of the interaction between radiation and matter. Jordan used the new statistics, as well as de Broglie's idea of matter waves to which Einstein had referred in order to motivate it, to study the thermodynamical equilibrium of light quanta and the ideal gas. This led him to make a strikingly novel stipulation:

> "The elementary acts of dispersion [between radiation and matter] can be viewed not only as dispersion of light radiation on material corpuscles but also as dispersion of matter radiation on corpuscular light quanta; therefore, the probabilistic law will be symmetric ... [between the densities of radiation and matter]."[3]

Schrödinger had taken Einstein's theory of the ideal gas as evidence that matter and radiation both had to be understood as waves [Schrödinger, 1926b]. Jordan agreed that matter and radiation were of the same nature, but he did not accept that this nature was correctly expressed by a classical wave picture. Instead, he postulated that both matter and radiation should be representable equivalently either as waves or as particles, thus establishing a complete symmetry between the two representations.

In an interview with Thomas Kuhn for the Archives for the History of Quantum Physics (AHQP),[4] Jordan credited the idea of the symmetry

[2] Published as [Jordan, 1924].
[3] Jordan 1925
[4] Interview of Pascual Jordan with Thomas S. Kuhn, June 18, 1963. AHQP, Transcripts of Oral History Recordings, Microfilm 1419-03, Jordan interview 2, p. 19.

of representations to William Duane's treatment of the scattering of light quanta by a grid [Duane, 1923]. Duane had shown that the interference on a grid, which had always been seen as a paramount wave phenomenon, could also be explained in the light quantum theory if one quantized the periodic structure of the grid. Jordan saw this argument as evidence that the dualism of particle and wave character of light should find its theoretical expression in the possibility to represent the same physical situation equivalently in particle and in wave description. For Jordan, this symmetry of representations was a convincing argument that all previous mechanical pictures had to be insufficient. The symmetry of representations would become the fundamental heuristic principle underlying Jordan's work both in quantum mechanics and quantum field theory during the following years. Jordan claimed in the AHQP interview[5] that already at this point he was hoping that a quantum theory of waves could deliver this symmetrical representation for both matter and radiation. Although there is no direct contemporary evidence, the circumstances described above make this plausible.

In the summer of 1925, Jordan got recruited by Max Born to help in the mathematical elaboration of Werner Heisenberg's idea of *Umdeutung*. Born and Jordan [1925] showed that the matrix calculus was the appropriate mathematical form for Heisenberg's new mechanics. However, Jordan did not limit himself to the formalization of Heisenberg's ideas: the paper contains an application of matrix mechanics to the electromagnetic field. This section did not lead to any concrete empirical predictions, and was largely ignored. But it is an indication of Jordan's program of a quantized field theory, rooted in his earlier insights from gas theory. Also the subsequent *Dreimännerarbeit* [Born et al., 1926] contains a section on the quantization of a field, this time with a much more striking result: the derivation of Einstein's famous and puzzling fluctuation formula for radiation from matrix mechanics applied to a field. As we know from a letter from Heisenberg to Pauli,[6] it was written

[5] Interview of Pascual Jordan with Thomas S. Kuhn, June 19, 1963. AHQP, Transcripts of Oral History Recordings, M/f 1419-03, Jordan interview 3, p. 9.

[6] Heisenberg to Pauli, October 23, 1925 [Pauli, 1979, p. 252].



by Jordan who later considered it as "almost the most important thing I have contributed to quantum mechanics."[7]

Einstein had used the thermodynamic entropy of radiation to derive its fluctuation properties: the energy fluctuations in a small volume of a small band of frequencies contained two terms. One could be interpreted as expressing fluctuations due to a varying number of light quanta in the volume, the other as due to the interference of light waves. Their simultaneous presence was a striking illustration of the dual nature of light but also posed the problem to find a theory of light that could account for the presence of both terms. Einstein struggled for the rest of his life to provide such a theory of light. In a study of Einstein's fluctuation formula, Paul Ehrenfest [1925] had introduced the model of a vibrating elastic string fixed at both ends as the simplest possible situation for the study of wave fluctuations. Each characteristic frequency of its vibration (or wave mode) can be treated as an independent harmonic oscillator. The total energy of each mode (and thus of the string as a whole) is constant. But the energy content of a small number of neighboring wave modes in a small segment of the string fluctuates because of the interference of the neighboring wave modes. Ehrenfest calculated this fluctuation and obtained only the wave fluctuation term, even if the individual wave modes were quantized in the sense of the old quantum theory. In the *Dreimännerarbeit*, Jordan quantized Ehrenfest's model using matrix mechanics – harmonic oscillators being one of the few things one could quantize with matrix mechanics in 1925 – and discovered that the non-commutativity of the matrix calculus leads to an additional term for the energy fluctuations: it is exactly the particle fluctuation term. For the first time, Einstein's fluctuation formula had been derived from an underlying dynamical theory.

Jordan concluded his considerations with the remark:

> "If one considers that the question treated here [the fluctuation of radiation] is rather removed from the problems out of which quantum mechanics arose, one will perceive the result [...] as especially encouraging for the further extension of the theory."[8]

[7] Jordan to van der Waerden, April 10, 1962, AHQP M/f 1419-006, p. 604. The quotes from Heisenberg and Jordan are given in [Duncan and Janssen, 2008].

[8] Born et al. 1926, p. 615

The full meaning of this remark would have eluded a contemporary reader, but it fits very well with Jordan's later reminiscences that he saw in this derivation the first lead to the quantized field theory he had been looking for. However, even his coauthors Heisenberg and Born were skeptical about the need to quantize the electromagnetic field [Duncan and Janssen, 2008, p. 640–642]. One obvious problem was that Jordan's method implied that each of the quantized oscillators representing the radiation field had a zero-point energy, so that the vacuum had an infinite energy density. This led Heisenberg to state that the method is only suitable for the treatment of oscillations of a discrete crystal lattice where such infinities would not occur. Jordan, on the other hand, had an even more ambitious goal: His principle of symmetries of representations implied that also matter should be represented by quantized waves in the same manner. As he claimed in [Jordan, 1927g, p. 480] and in a letter to Schrödinger, his occupation with the quantum theory of the ideal gas had suggested this further application of the theory of quantized waves. Jordan writes in the letter:

> "Then your hydrogen paper [i. e., Schrödinger [1926a]] gave hope that by following up this correspondence also the non-ideal gas could be represented by quantized waves – that therefore a complete theory of light and matter could be derived in which, as an essential ingredient, this wave field itself operates in a quantum, non-classical way."[9]

Jordan saw Schrödinger's wavefunctions as a generalization of the simple plane waves that he had quantized in the *Dreimännerarbeit* and interpreted as the quantum mechanical representation of the Bose-Einstein ideal gas; he was convinced that the quantization of these wavefunctions was the method necessary to apply quantum mechanics to the case of several interacting particles.[10] In the letter to Schrödinger, Jordan gives two reasons why he did not pursue this program immediately: The problem to account for Fermi-Dirac statistics, since it seemed that the wave picture would always lead to Bose-Einstein statistics, and the reservations of his colleagues Heisenberg, Pauli, and Born.

[9] Jordan to Schrödinger, reply to Schrödinger's letter from July 28, 1927, AHQP M/f 41 Sect. 8-009b, quoted after Darrigol, p. 224.

[10] Interview of Pascual Jordan with Thomas S. Kuhn, June 20, 1963. AHQP, Transcripts of Oral History Recordings, M/f 1419-03, Jordan interview 4, p. 3.



By the summer of 1926, Jordan was thus convinced that the correct treatment of a system of interacting particles was the quantization of their associated matter waves. This approach was fundamentally different both from Schrödinger's and from Heisenberg's and Dirac's ideas about the application of quantum mechanics to the many-particle problem. While Schrödinger was searching for a way to represent the many-body problem as the self-interaction of a continuous charge distribution, Heisenberg and Dirac had constructed symmetrical and antisymmetrical many-particle wavefunctions from single-particle wavefunctions and given phenomenological arguments why they should account for the characteristics of atomic spectra. Dirac showed that symmetrical wavefunctions led to Bose-Einstein statistics and that antisymmetrical wavefunctions explained the Pauli exclusion principle for electrons and therefore should be the basis of a statistics for matter particles. The success of the Heisenberg-Dirac method in the explanation of atomic spectra made Jordan's much more abstract program seem superfluous.

The transformation theory, developed in 1926/27 by Dirac [1927a] and Jordan [1927e, f] independently, was for Jordan further evidence for his principle of symmetry of representations. To Jordan, it showed that there is no preferred ontological basis in which quantum mechanics should be explicated. Jordan's transformation theory did not use the concept of a state at all; rather, what he used for the description of a physical system was the totality of all possible transition amplitudes between the values of physical quantities, the squares of which give the probability of finding the value of one quantity given the value of another quantity. Instead of specifying, e. g., one specific state of a hydrogen atom by a wavefunction, Jordan's transformation theory describes all possible states of the hydrogen atom by the transition amplitudes between a basis diagonalizing the energy matrix and a basis diagonalizing the position matrix of the electron. Jordan now identified "particle" properties with the basis diagonalizing the position matrix and "wave" properties with the basis diagonalizing the momentum matrix conjugate to the position matrix. Since the theory is invariant with regards to the choice of basis, the system can be described equally in particle or wave language. Therefore, neither description of the system (as a particle or as a wave) is fundamental.

This conviction about the symmetry of representations was also the background for Jordan's attack on Schrödinger's physical wave interpretation of wave mechanics [Jordan, 1927d]. Jordan agreed with Schrödinger that light and matter show analogous behavior and should be treated analogously in quantum theory. But he argued that just as classical wave optics fails for the effects that made the light quantum theory necessary, so wave mechanics alone cannot account for the particulate aspects of matter. Otherwise, there would be a disanalogy between the theories of light and matter.

## 3 The beginning of quantum field theory

The idea of a quantized field only came to the attention of a wider group of physicists through Paul Dirac's "The quantum theory of emission and absorption of radiation."[11] Paradoxically, the notion of quantizing a field appears nowhere in the paper. Dirac started with standard perturbation theory and observed that the expansion of the perturbed state $\psi$ in terms of the eigenstates $\psi_r$ of the unperturbed Hamiltonian $H_0$

$$\psi = \sum_r a_r \psi_r \tag{1}$$

can be interpreted as describing how a statistical ensemble of noninteracting systems reacts to an external perturbation, since the squared expansion coefficients $|a_r|^2$ can be read as giving the ratio of systems in each eigenstate. Standard perturbation theory gives for a perturbed Hamiltonian $H = H_0 + V$ the following time-dependence of the expansion coefficients:

$$i\hbar \dot{a}_r = \sum_s V_{rs} a_s \tag{2}$$

Dirac now showed that if one treated the $a_r$ as quantum numbers, the same equations can be interpreted as describing an ensemble of systems obeying Bose-Einstein statistics. In this case, $N_r = a_r^\dagger a_r$ gives the number of systems in state $r$. If one applies this procedure to a system of light quanta interacting with an atom, one can represent the interaction in

[11] Dirac 1927b



terms of the changes that it causes in the atomic states and in the number of light quanta.

Dirac never tried to relate the $a_r$ directly to field amplitudes. Rather, he connected the two by observing that a given number of light quanta determines through Einstein's $E = h\nu$ the energy density of the corresponding electromagnetic field and thus the field amplitudes acting on the atom. Using this equation, he could connect Einstein's emission and absorption coefficients with the matrix elements of the atomic electron in matrix mechanics – something that Heisenberg had only postulated in the *Umdeutung* paper. However, Dirac explicitly denied that the "wave function of the light quanta" is the same as the electromagnetic field. He also argued that while an ensemble of light quanta can be associated with a light wave, there is no such physical wave associated with an ensemble of matter particles such as electrons. Therefore, he did not see the quantization procedure as an explanation of the quantum nature of radiation. It was to him only an elegant way to take into account the Bose statistics of light quanta. Since electrons do not obey Bose statistics, the procedure is not applicable to them. Dirac maintained particle number conservation for light quanta by introducing a 'sea' of zero-momentum light quanta. This is another piece of evidence that for Dirac the particle concept was primary.[12]

Unlike Jordan's earlier attempt, Dirac's theory was greeted with enthusiasm, since it first derived the link between quantum mechanics and Einstein's theory of absorption and emission, and so offered a quantum-mechanical representation of the interaction of matter and radiation. Today, Dirac's paper is often seen as the seminal work for quantum field theory. This is somewhat ironic, as Dirac explicitly rejected the idea that his method was to be understood as the quantization of the classical field. Jordan thought for the rest of his life that he did not get due credit for his work:

> "It has always saddened me somehow that the attack on the light-quantum problem already contained in our Dreimännerarbeit was rejected by everyone for so long (I vividly remember how Frenkel, despite his very friendly disposition toward me,

[12] It also shows the problems that interpreting light quanta as particles leads to, foreshadowing the even more problematic notion of a sea of negative-energy electrons that would appear in Dirac's 1928 electron theory.

regarded the quantization of the electromagnetic field as a mild form of insanity) until Dirac took up the idea from which point onward he was the only one cited in this connection."[13]

Instigated by Dirac's success, Jordan quickly returned to the theory of the quantized field. However, what he did was in conflict with Dirac's ideas and a clear continuation of his earlier program based on the principle of symmetry of representations. Therefore, his first paper [Jordan, 1927g] explicitly rejected Dirac's assessment that the ideal gas obeying Fermi statistics cannot be represented by a wave field. Jordan observed that in the case of Bose-Einstein statistics, the number operator has arbitrary integer eigenvalues, while in the case of Fermi-Dirac statistics, the number operator can only have eigenvalues 0 or 1. He now constructed an algebra of field operators that yield these eigenvalues for the number operator using Pauli's spin matrices. This construction was made possible by Jordan's concept of conjugate variables that was more general than Dirac's: While Dirac relied on commutation relations of the standard form $pq - qp = -i\hbar$, Jordan's transformation theory relied on a more general notion of conjugate variables (motivated by the need to represent angle and angular momentum as conjugate variables[14]) and allowed for a generalization of these commutation rules. However, as Darrigol [1986, p. 232] has pointed out, Jordan's actual calculations were full of mistakes: "Although Jordan knew he was on the right track, his paper was only a sketch, full of misprints and imprecisions. The draft received by Alfred Landé resembles a bad student paper overcorrected by the professor." What had gotten lost in the imprecisions were the correct phase relations between the creation and annihilation operators. Only in the fall of 1927, Jordan would return to the topic and, with the help of Eugene Wigner, present the correct algebra (now called Jordan-Wigner second quantization) using anticommutation relations [Jordan and Wigner, 1928].

Despite its technical flaws, [Jordan, 1927g] already defines Jordan's program: a unified quantum field theory for matter and radiation.

[13] Jordan to Born, July 3, 1948, AHQP M/f 1419-006, p. 596; quoted after Duncan and Janssen, 2008

[14] Interview of Pascual Jordan with Thomas S. Kuhn, June 19, 1963. AHQP, Transcripts of Oral History Recordings, M/f 1419-03, Jordan interview 3, p. 22–23.



Particles and waves are only two different aspects of the same underlying quantum field both in the case of light and in the case of matter:

> "Despite the validity of the Pauli instead of Bose statistics for electrons, the results achieved so far leave hardly a doubt that a quantum-mechanical wave theory of matter can be formulated, in which electrons are represented as quantized waves in ordinary three-dimensional space and that the natural formulation of the quantum theory of the electron will have to be achieved by comprehending light and matter on equal footing as interacting waves in three-dimensional space. The fundamental fact of electron theory, the existence of discrete electrical particles, thus manifests itself as a characteristic quantum phenomenon, namely as equivalent to the fact that matter waves only appear in discrete quantized states."[15]

Jordan pointed out that the antisymmetrical wavefunctions that Heisenberg and Dirac had constructed for many-particle systems were therefore not at all physical waves but simply "a special case of the general probability amplitudes which have to be used as a mathematical tool for the description of the statistical behavior of quantized light and matter waves" [Jordan, 1927g, p. 480]. These quotes show clearly the difference in perspective between Jordan and Dirac: Unlike Dirac, Jordan treated second quantization of the Schrödinger wave function as the quantization of a physical field and saw this procedure as an explanation of the corpuscular character of matter. Unlike Schrödinger, however, Jordan did not attempt to find an objective physical description behind the mathematical formalism. Transformation theory to him still implied that neither the particle nor the wave description were fundamental and therefore neither picture could be used to construct a complete description of objective reality.

Jordan's vision was not yet a full theory. So far, he only could treat free fields nonrelativistically. In the following months, Jordan made quick progress towards a complete theory in a series of three papers with different collaborators. First, he collaborated with Wolfgang Pauli [Jordan and Pauli, 1928], giving relativistically invariant commutation rules for the free field. The second paper was written together with Oskar Klein in Copenhagen [Jordan and Klein, 1927]. Klein had been thinking

[15] Jordan 1927g, p. 480

about a relativistic quantum theory of interacting particles, based on Dirac's quantized waves. As he wrote to Dirac, he worried about the problem of self-energies arising from the field-theoretical treatment of the Coulomb interaction.[16] When Jordan stayed in Copenhagen in the summer of 1927, they introduced field operators $\phi(r)$ to represent the field strength in a specific spacetime point and solved the problem of self-energies by what is now called normal ordering of these field operators. This allowed for a quantum field theoretical reformulation of the (instantaneous) interaction between particles and demonstrated that quantum field theory can treat the many-particle problem, as Jordan had envisioned already in 1926.

Schrödinger, referring to the programmatic passage from [Jordan, 1927g] cited above, wrote to Jordan in surprise:

> "This is, as far as I understand, also my opinion. So far, I thought that it was decidedly rejected from Göttingen and Copenhagen. Now I am glad to see that prospects are improving that we will come together again."[17]

Born, Heisenberg, and Pauli referred to Jordan's work at the Solvay meeting in October of 1927, as a possible solution to the problems faced when explaining quantum effects based on a wave picture. Also Bohr was impressed and praised the work by Jordan and Klein in [Bohr, 1928]. Dirac, however, was not convinced and, in the discussions at the Solvay meeting (yet never in writing), criticized Jordan's quantization procedure as artificial and *ad hoc*. He also pointed out that there were mistakes in the mathematical derivation of [Jordan, 1927g]. When in 1928 Dirac developed his relativistic theory of the electron [Dirac, 1928], he treated the relativistic wave equation as an analogue of the Schrödinger equation and did not make use of any field-theoretical interpretations.

The first attempt at a full treatment of quantum electrodynamics was given by Heisenberg and Pauli [1929]. But this treatment also showed the problems connected with the quantum field theory program. As Jordan [1929] noted, the infinite self-energy of the electron was not a constant that could be simply ignored as in the case of the free

[16] See [Darrigol, 1986, p. 234].

[17] Letter from Erwin Schrödinger to Pascual Jordan, 28 July 1927, AHQP, M/f 18, Sect. 7-001.



field. Jordan remarked that this problem was inherited from classical electrodynamics and that therefore it showed the limitations of the procedure of quantizing a classical theory. A new autonomous quantum field theory would have to be found to solve the problem. While various proposals to remove the infinities were made in the following years, it remained unclear how a general theory without inconsistencies could be built up. Possibly even more damaging to the program was the fact that it did not offer empirical predictions going beyond a theory such as Dirac's treating particles with antisymmetrical wave functions.

Only after World War II did the observation of the Lamb shift offer a first empirical confirmation of vacuum fluctuations, leading to a resurgence in interest in the quantum field theory program.[18] Quickly, this led to new renormalization techniques and the successful treatment of perturbation theory with Feynman diagrams. Jordan and Dirac, however, never rejoined the forefront of research in quantum field theory. Jordan's early contributions were mostly forgotten by the time of the postwar renaissance of quantum field theory, even though its modern formulation is closer to Jordan's program than to Dirac's original ideas.

## 4 Positivism

Schrödinger's hope for a reapproachment between his views and those of Jordan was not shared by the latter. Despite Jordan's polite answer, there was no indication that Jordan was changing his views already expressed in connection with transformation theory, that quantum mechanics did not allow for a reduction to classical models, be they particles or waves. As he would state in 1936 in his programmatic popular account "Physics in the 20$^{th}$ century":

> "The atom as we know it today no longer has the tangible and visualizable properties of the atoms of Democritus. It has been stripped of all sensible qualities and can only be characterized by a system of mathematical equations. The unbridgable opposition of materialistic philosophy and positivistic epistemology stands out especially clearly at this point. With this insight, one of the

[18] See [Schweber, 1994] for a treatment of the history of quantum electrodynamics after World War II.

most prominent elements of the materialist world view has been liquidated once and for all. At the same time, the positivistic epistemology has been confirmed and justified decisively."[19]

The fundamental lesson Jordan drew from quantum physics was a confirmation of positivism. The basis for this bold metatheoretical claim[20] was Jordan's conviction that the symmetry of different descriptions established by transformation theory implied that there was no *one* fundamental physical description and that therefore statements about unobservable entities in quantum mechanics (which corresponded to one specific description, i. e. the wave or particle picture) were meaningless.

However, neither Jordan's positivism nor his argument for it from transformation theory harmonize very well with his program for quantum field theory: Jordan's claims about the foundational character of quantum field theory imply a priority of an abstract field concept, with particles as secondary quantum phenomena. This abstract field concept, even though it does not coincide with Schrödinger's more physical concept of a matter field, retains one important characteristic of the classical field: the continuity and classical description of spacetime. No matter what representation is chosen, the states of the theory are defined on this continuum. For that reason, transformation theory does not have the same implications in quantum field theory as it does in quantum mechanics. Even though Jordan is not explicit about how he understands the application of transformation theory to quantum field theory, he seems to assume that particle and wave properties are represented by the two basic quantities of his formalism, the $\phi(r)$, describing the field strength at the position $r$, and the $b_k$, describing the amplitude of the excitation with the wavevector $k$.[21] Although these two quantities are related by a Fourier transform

$$\phi(r) = \sum_k b_k u_k(r) \tag{3}$$

(which resembles the Fourier transform between position and momentum eigenstates in quantum mechanics), this does not mean that $\phi(r)$ can be identified with a particle property (i. e., a particle being in the position $r$). $\phi(r)$ only specifies the field strength at the position $r$, not a localization of the field at $r$. In Jordan's terminology: The matrix $\phi(r)$ is highly degenerate and therefore does not specify a basis that suffices to describe *localized* excitations of the field. Therefore, the Fourier transform is not the formal expression of a symmetry between wave and particle representations, unlike in the case of quantum mechanics. Thus, Jordan's quantum field theory is not symmetrical between wave and particle representations and so does not confirm positivism in the same way that he believed transformation theory did. Rather, one could say that wave and particle picture are represented by Jordan's field theory and Dirac's "many-particle theory" of symmetrized or antisymmetrized wave functions. But these are two distinct theories, which only coincide in certain cases.[22]

More generally, one can observe that Jordan's grand foundationalist visions are at odds with his positivism: According to the 19$^{\text{th}}$ century understanding of positivism, physical theory should describe, not explain. But Jordan himself kept invoking the explanatory power of quantum field theory as a justification of its fundamental nature, e. g., in the above quote from [Jordan, 1927g, p. 480]. Despite these tensions, Jordan maintained his positivism by emphasizing the differences between his quantized fields and classical fields. He frequently stressed that the quantum field did not offer hope for picturability in the classical sense. Therefore Jordan could maintain that, although quantum field theory offers a unified foundation of physics, it does not offer a visualizable physical model of the world. All it provides are probability amplitudes connecting possible observations, like in the case of transformation theory. However, this is a much weaker argument than in the case of the explicit argument for the possibility of different representations – it does not exclude the possibility that a *non-classical* but still spatiotemporal field picture could eventually be found as a consistent model for quantum field theory.

19 Jordan 1936, pp. 122 – 123

20 See [Jordan, 1934] for a defense of positivism as a general epistemological principle, and [Darrigol, 1986, pp. 232 – 233], [Cini, 1982] for discussions of Jordan's positivism.

21 In modern terms, these are the field operator and the annihilation operator, respectively.

22 A simple aspect in which they do not coincide is that for Dirac, particle number must be conserved, while for Jordan, this is not necessarily the case.

## 5 Mathematics and physics

It is somewhat difficult to define the relation between mathematics and physics in Jordan's work. Exactly because of his Göttingen background he does not seem to see the two as distinct research subjects. In the AHQP interviews, he characterizes himself as a «Göttinger» in several places, contrasting his own open-minded attitude towards mathematical formalism to the suspicion if not hostility towards it from other physicists. For example, he relates the well-known story that Pauli accused Born that he would mess up Heisenberg's «Umdeutung» ideas with excessive mathematical formalism. However, and this is the more important observation, Jordan goes beyond what was traditionally seen as the role of a mathematical physicist: the precise elaboration of existent physical theories (say, in analytical mechanics). This difference becomes quite evident in comparison with John von Neumann's work, and his central contribution to quantum physics, the introduction of the Hilbert space formalism. Von Neumann's formalization gave a firm mathematical grounding to transformation theory, avoided Dirac's "improper functions," and allowed for new important concepts and arguments on the basis of the formalized theory, such as projection and density operators, quantum logic, his no-hidden-variable proof, or the formulation of the measurement problem. For all his important contributions, von Neumann's ambition was not to establish a new theory, but to clarify the existing statistical transformation theory. Thus his work is much more easily understood in the traditional sense of mathematical physics.

Jordan's strength, on the other hand, was definitely not the clarification of formal structures. We encountered a striking example above: Jordan needed Wigner's help to formulate the correct commutation relations for fermion fields. Another example is Jordan's half-hearted and confused attempt to present [Jordan, 1927e, f] in axiomatic form. Rather, Jordan's strength laid in his novel and far-reaching ideas about the foundations of quantum physics. In this respect, he was much more in the tradition of the previous generations of theoretical physicists, such as Planck, Einstein, and Schrödinger. Like these, he had the ambition to develop new and fundamental theories encompassing hitherto disjoint phenomena and the talent to find the correct clues in an abundance of



experimental data. An indication of his claim to the status of a theoretical physicist are his lucid review papers[23] and his eloquent presentations to general audiences (e. g. [Jordan, 1927a, b, c] in *Die Naturwissenschaften*).

However, there is a fundamental divide between Jordan and Einstein or Schrödinger. What Jordan sees as a proof of positivism from quantum physics is for them a *reductio ad absurdum* of quantum mechanics as a physical theory. Although they disagreed in their specific criticisms and their hopes for a better theory, they agreed in one point: The inability of quantum mechanics to produce unambiguous spatiotemporal models of objective processes disqualified it as a fundamental physical theory.[24] The disagreement about positivism was not merely a philosophical debate disconnected from physical theorizing, it fundamentally affected the definition of theoretical physics itself and its methodological prescriptions.

The central role of (mechanical) models for the foundations of classical theoretical physics is a well-treated subject.[25] I will only touch on one aspect that throws an interesting light on the relation of theoretical physics to mathematics: Elizabeth Garber contrasts the work of Poincaré as a mathematician in electrodynamics with that of Einstein and Lorentz as theoretical physicists. She notes that Poincaré had a different interest in exploring electrodynamics: "Poincaré's net was mathematics and observation, not physical theory."[26] This led Einstein to explore the physical consequences of Lorentz invariance, which Poincaré didn't. Einstein in turn did not see the relevance of Minkowski's geometrical representation of the Poincaré group, until his work on general relativity forced him to deal with it. One can therefore see the distinction between mathematics and theoretical physics in the focus on theoretical models

[23] E. g. [Jordan, 1928, 1929]

[24] The philosophical principles underlying Schrödinger's critique of quantum mechanics are treated in [Bitbol and Darrigol, 1992], Schrödinger's defense of the need for visualizability of physical theories in [De Regt, 1997]. In the case of Einstein, the existing literature is far too extensive to be cited in detail here. See [Home and Whitaker, 2007] for an overview; I will discuss Einstein's critique of positivism in quantum mechanics in a forthcoming contribution to *The Cambridge Companion to Einstein*, M. Janssen and C. Lehner, eds. Cambridge University Press, Cambridge.

[25] See for example [Lützen, 2005] for the case of Heinrich Hertz, [De Regt, 1999] for the case of Boltzmann, or [Cat, 2001] for the case of Maxwell.

[26] Garber 1999, p. 354

of physical situations and their exploration. While theoretical physicists took them as an expression of fundamental physical principles, mathematicians treated them as secondary illustrations of the fundamental mathematical structure.

This distinction connects the issue of positivism with the demarcation of theoretical physics from mathematics: Pauli, Heisenberg, and Jordan saw matrix mechanics as expressing the impossibility to give a consistent physical picture to quantum mechanical processes and Heisenberg's uncertainty relations as numerical limit to the applicability of such pictures.[27] As we saw, Jordan maintained this position also for the quantized field and was the most explicit in connecting it to a emphatic defense of positivism: Physics is about nothing but a concise mathematical description of the phenomena. Every question going beyond that is a pseudoproblem. The conspicuous absence of the concept of a physical state in Jordan's formalism reflects his conviction that there are no matters of fact beyond the observational data. According to Garber's distinction, his positivism therefore would make him a mathematician rather than a theoretical physicist – or at least it would have done so around the turn of the twentieth century. This verdict would have probably been applauded by Einstein and Schrödinger, who maintained that giving up a fundamental physical picture for quantum theory meant abandoning the core element of physical theorizing. And both, in different ways, kept fighting to regain such a physical picture.

However, as we have seen, this verdict is rather one-sided. It does not do justice to the relevance and foresight of Jordan's vision for quantum field theory. It is not that Jordan was not a theoretical physicist, rather theoretical physics changed radically in the years between 1900 and 1930. But there is something particular about Jordan's quantum field theory that put it in a precarious situation: Not only had Jordan abandoned the theoretical models of old, he also did not have a solid mathematical foundation at the basis of his theory. And this might very well be the reason why the pursuit of his theoretical visions was rather short-lived. When the problems of infinities in the Heisenberg-Pauli theory convinced Jordan that a straightforward quantization of Maxwell-Lorentz electrodynamics was not possible,

[27] See [Hendry, 1984] for a detailed discussion.



and that "radical new ideas" were necessary, he had no firm ground from which he could have kept trying. It is striking how quickly Jordan abandoned his program after 1929: No direct continuation of his work on the foundations of quantum field theory exists. Rather, in the following years, he turned to biology, to mathematics, and to the right-wing politics that should permanently damage his reputation. Only in the mid-thirties there were some unsuccessful attempts to resuscitate his work on quantum field theory. Unlike Einstein and Schrödinger who kept developing their chosen models, despite success kept evading them, and against the opinion of the mainstream, Jordan had no fundamental structure to fall back on in the face of his problems.

## 6 Epilogue

It was not just Jordan, but his whole generation that rejected the idea of theoretical physics that Einstein and Schrödinger defended. The triumph of quantum mechanics convinced theoretical physicists ever since that they could do their job without recourse to visualizable models. But unlike in the case of quantum mechanics, where von Neumann's Hilbert space formalism offered a clear and solid mathematical foundation, quantum field theory up to this day has not been cast into a definite mathematical structure. The algebraic approach has been an attempt in that direction, but has not yet arrived at a point where it successfully reconstructs the theory that physicists use.

In its physical foundations, on the other hand, modern-day quantum field theory equally suffers from lack of clarity and definiteness. Just as in the days of Dirac and Jordan, it sometimes is interpreted as field theory, sometimes as particle theory; its proudest technical achievement, renormalization theory, lacks a physical interpretation or theoretical justification. Nor is there a clear account of the relation of quantum field theory to quantum mechanics, its nonrelativistic limit. In their daily work most physicists use varying visualizations, especially Feynman diagrams, as a substitute for physical models. But they are quite aware that their use is very limited and in the end they just rely on pragmatic rules when the model fails. Just like Jordan, quantum field theorists today still have a grand vision of a unified theoretical framework for all of physics. But (as Einstein might add) just like him they are still suspended in a no-man's land between physics and mathematics.

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