The analyticity principle in 20th-century physics: from the Kramers dispersion formula to dual amplitudes and strings

Abstract

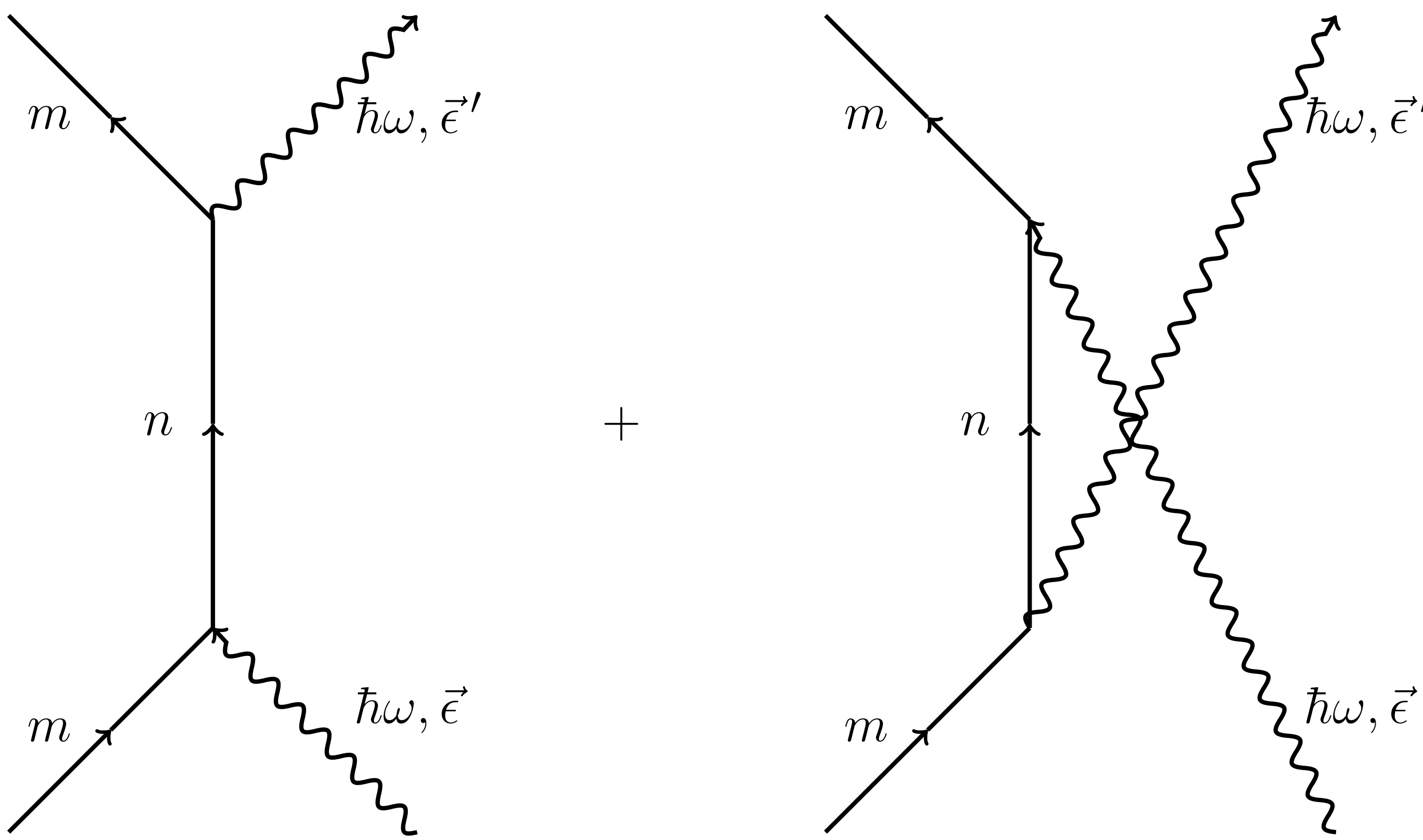
Two formulas of great historical resonance in twentieth century theoretical physics- the Kramers dispersion formula of 1924, the immediate precursor of Heisenberg's matrix mechanics, and the Veneziano dual amplitude of 1968, the direct ancestor of the Nambu-Nielsen bosonic string- are continuously connected by a conceptual strand of enormous importance. The requirement of analyticity of quantum-mechanical amplitudes, intimately related to the demands of causality, as a constraint on fundamental physics, is perhaps second only to the emergence of the local gauge symmetry principle in guiding the development of modern microphysical theories from relativistic quantum field theory to string theory. In this talk, some of the important signposts along the path connecting these two seminal formulas will be discussed.

Seven Pines Symposium XVII; May 16-19 2013

Two remarkable formulas with seminal consequences for twentieth-century physics:

- Kramers dispersion formula for light scattering from bound electrons (direct precursor to Heisenberg's matrix mechanics)
- Veneziano formula for high-energy meson scattering with duality symmetry (direct precursor to hadronic string theory)

Elastic Light Scattering from Atoms-
the Kramers dispersion formula (1924)

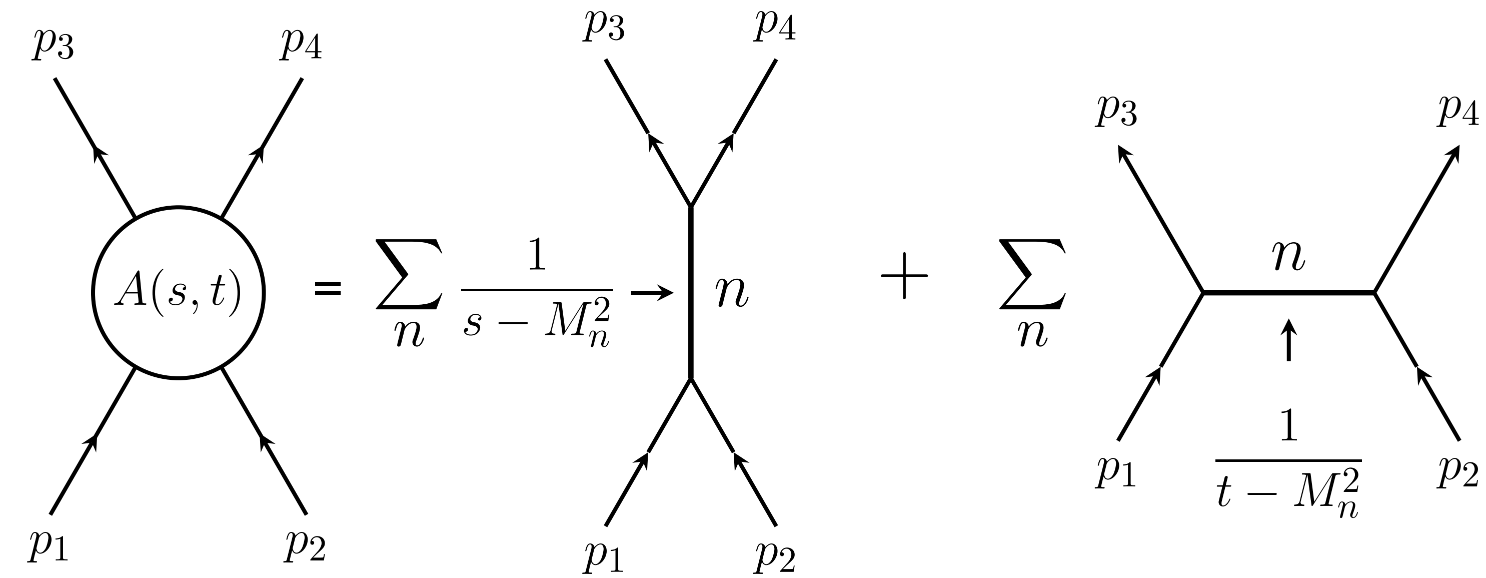


$$\mathcal{M}(\omega) \propto \sum_n \left\{ \frac{(\vec{\epsilon}\,' \cdot \vec{X})_{mn}(\vec{\epsilon} \cdot \vec{X})_{nm}}{E_m - E_n + \hbar\omega} + \frac{(\vec{\epsilon} \cdot \vec{X})_{mn}(\vec{\epsilon}\,' \cdot \vec{X})_{nm}}{E_n - E_m - \hbar\omega} \right\}$$

... formulae which contain only the frequencies and amplitudes which are characteristic for the transitions, while all those symbols which refer to the mathematical theory of periodic systems (i.e. Bohr orbits) will have disappeared (Kramers and Heisenberg,1924)

Methodological Background- Bohr Correspondence Principle

High Energy Meson Scattering-
the Veneziano Model (1968)



field theory

$s = (p_1 + p_2)^2$ $\qquad$ $t = (p_1 - p_3)^2$

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}, \quad \alpha(s) = \alpha_0 + \alpha' s$$

Veneziano amplitude

$$A(s,t) \propto \sum_n \frac{(\alpha(t)+1)(\alpha(t)+2)\cdot\cdot(\alpha(t)+n)}{s-M_n^2}, \quad M_n^2 = -\frac{\alpha_0}{\alpha'} + \frac{1}{\alpha'}n$$

$$= \sum_n \frac{(\alpha(s)+1)(\alpha(s)+2)\cdot\cdot(\alpha(s)+n)}{t-M_n^2}$$

dual model

In dual models, the entire amplitude is given by either the s-channel or the t-channel resonance sums, but not both!

Methodological Background: the Bootstrap Principle, in principle determining masses and scattering amplitudes of hadronic resonances uniquely from analyticity, unitarity and crossing

The Kramers-Kronig Dispersion relation

Ref: *La diffusion de la lumiere par les atomes*, M.H.A. Kramers, Atti Congr. Intern. Fisici, Como, **2**, 545-557 (1927)
On the Theory of Dispersion of X-rays, R. de L. Kronig, Jour. Opt. Soc. America, **12**, 547-557 (1926)

Equation of motion for classical charged oscillator in external oscillating electric field

$$\ddot{x} + \Gamma \dot{x} + \omega_1^2 x = \frac{e\mathcal{E}}{m} e^{i\omega t}$$

Radiative decay constant $\Gamma > 0$ - radiation reaction due to retarded (no advanced) radiation. This is a *causality* requirement. Solution:

$$P(t) = ex(t)/\mathcal{E} = \zeta(\omega) e^{i\omega t}$$

with

$$\zeta(\omega) = \frac{e^2}{m} \frac{1}{\omega_1^2 - \omega^2 + i\Gamma\omega}$$

Note that the singularities (poles) of $\zeta(\omega)$ are *above the real axis*:

$$\omega = \frac{i\Gamma}{2} \pm \sqrt{4\omega_1^2 - \Gamma^2}$$

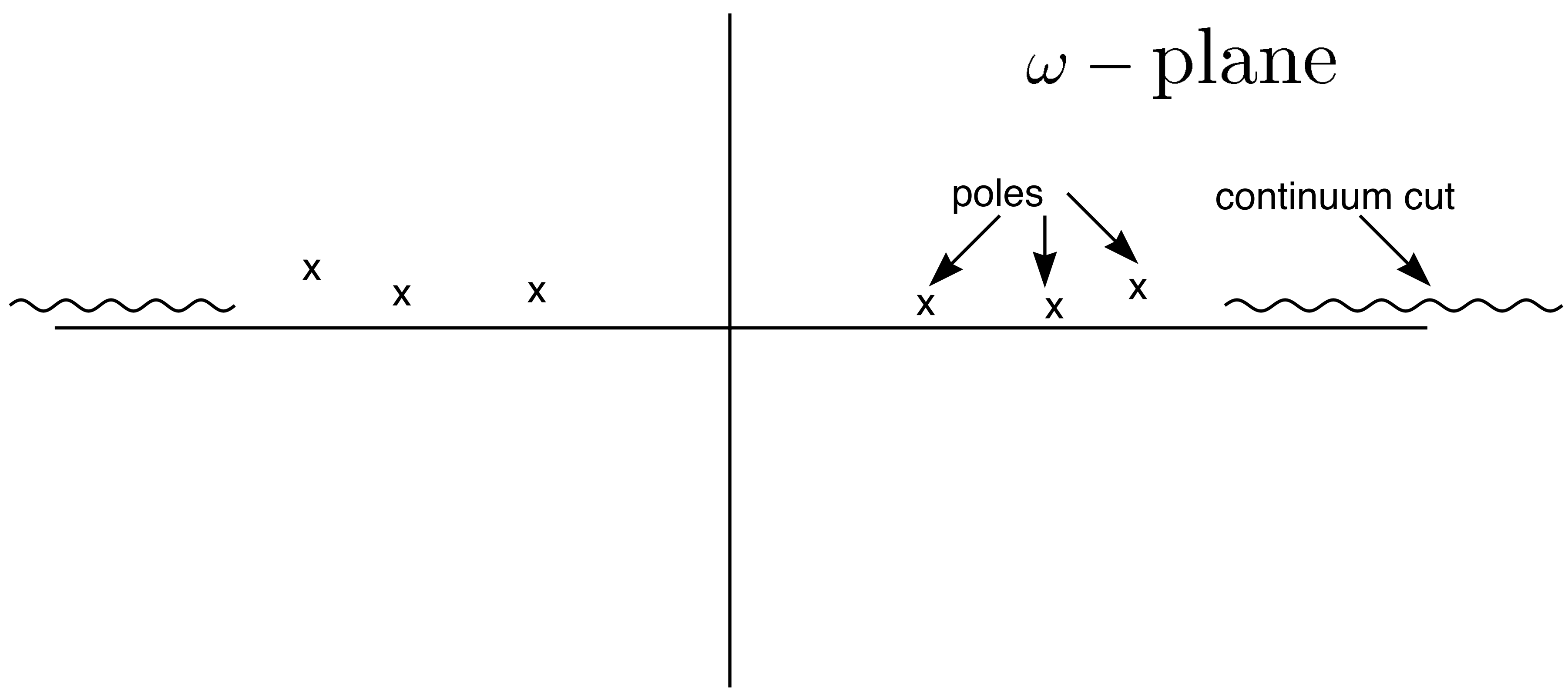
Ladenburg-Kramers generalization for atoms in their normal (ground state):

$$\zeta(\omega) = \sum_k \frac{e^2}{m} \frac{f_k}{\omega_k^2 - \omega^2 + i\Gamma_k \omega}$$

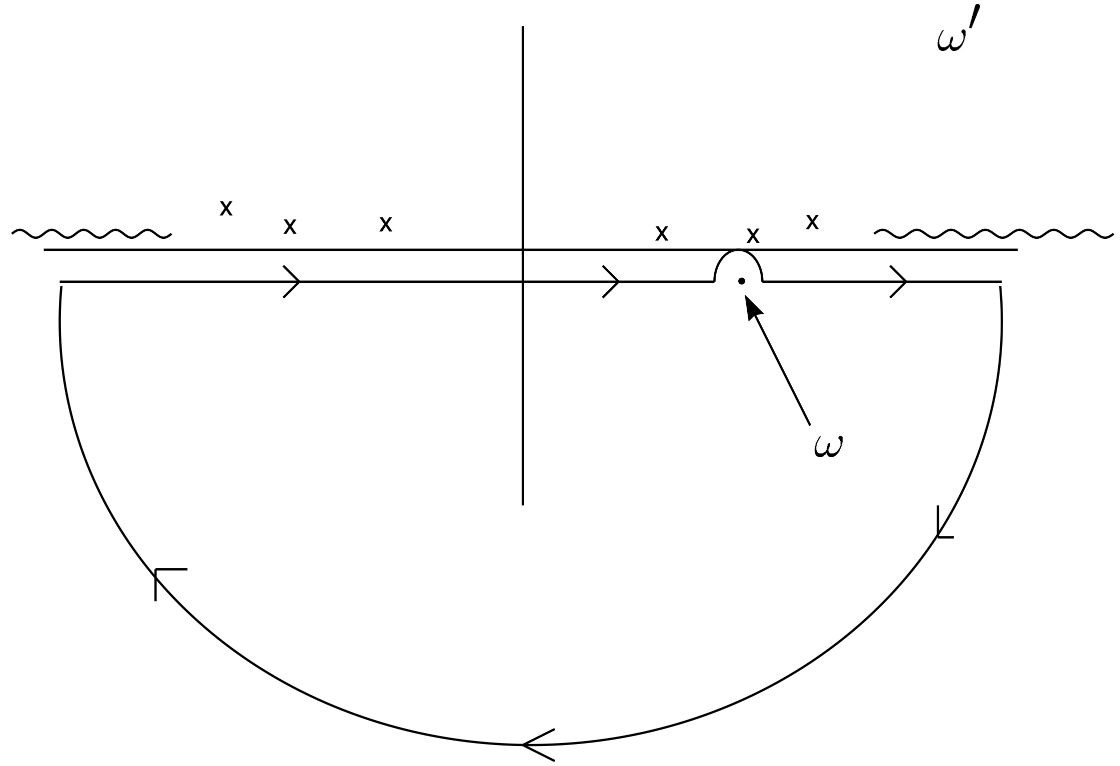
Note that although initially defined only for real $\omega > 0$, we can extend $\zeta(\omega)$ to negative (real) ω, with

$$\zeta(-\omega) = \zeta^*(\omega)$$

If we include the continuous spectrum, there is also a cut in the complex ω plane beginning at the ionization frequency. Viewing ζ as an analytic function in the complex frequency plane, we have the following situation-



⇒ lower half-plane analyticity



Lower half-plane analyticity allows the Cauchy representation

$$\begin{aligned}
\zeta(\omega) &= -\oint \frac{d\omega'}{2\pi i}\frac{\zeta(\omega')}{\omega'-\omega} \\
&= \frac{1}{2}\zeta(\omega) - \frac{1}{2\pi i}P\int_{-\infty}^{+\infty} d\omega' \frac{\zeta(\omega')}{\omega'-\omega} \\
&= \frac{1}{2}\zeta(\omega) - \frac{1}{2\pi i}P\int_{0}^{+\infty} d\omega' (\frac{\zeta(\omega')}{\omega'-\omega} + \frac{\zeta^*(\omega')}{\omega'+\omega})
\end{aligned}$$

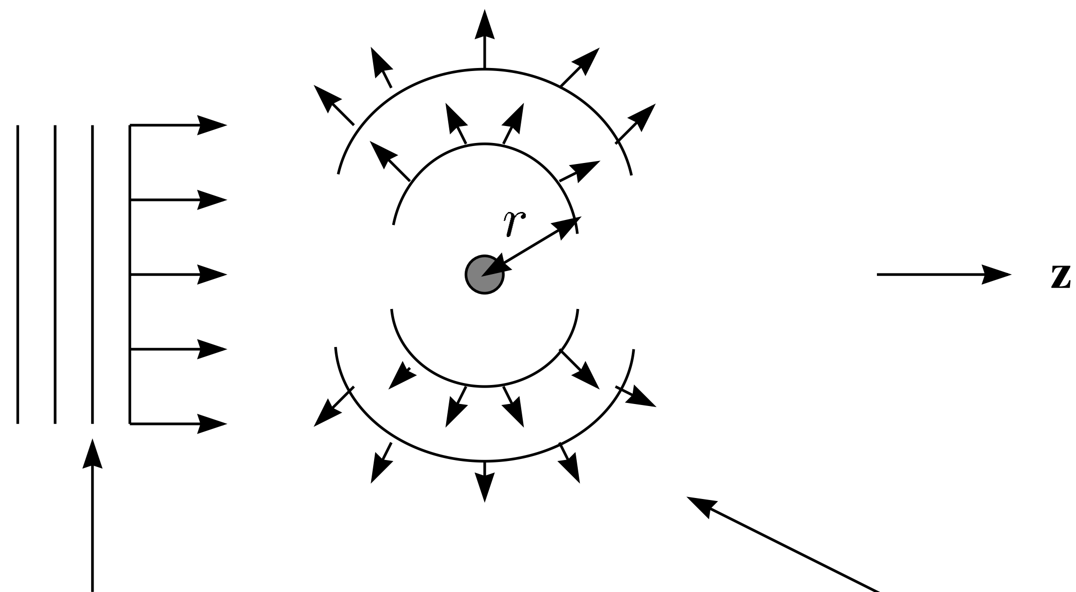
assuming that the large semicircular contour gives a vanishing contribution. Thus

$$\zeta(\omega) = \frac{1}{\pi i}P\int d\omega' (\frac{\zeta^*(\omega')}{\omega'+\omega} - \frac{\zeta(\omega')}{\omega'-\omega})$$

In terms of real (dispersion) and imaginary (absorption) parts, $\zeta(\omega) \equiv \xi(\omega) + i\eta(\omega)$:

$$\xi(\omega) \quad = \quad -\frac{2}{\pi}P\int d\omega' \frac{\omega'\eta(\omega')}{\omega'^2-\omega^2} \quad \text{Kramers (19)}$$

Causality $\Rightarrow$ Half-plane analyticity, a more general argument



$$\psi_{\mathrm{in}}(z,t) = g(\omega)e^{i\omega(\frac{z}{c}-t)}$$

$$\psi_{\mathrm{scat}}(z,t) = f(\omega)g(\omega)e^{i\omega(\frac{r}{c}-t)}$$

Form a wave packet of incoming waves:

$$\psi_{\mathrm{in}}(z,t) = \int d\omega g(\omega)e^{i\omega(\frac{z}{c}-t)}$$

The scattering amplitude $f(\omega)$ then gives the outgoing spherical scattered wave as

$$\psi_{\mathrm{scat}}(r,t) = \int d\omega f(\omega)g(\omega)e^{i\omega(\frac{r}{c}-t)}$$

Now choose $g(\omega)$ so the incoming packet arrives at the scattering center at $t = 0$:

$$\psi_{\rm in}(0,t) = 0, \quad t < 0, \quad \text{with} \;\; g(\omega) = \int_0^{+\infty} \frac{dt}{2\pi} \psi_{\rm in}(0,t) e^{i\omega t}$$

$g(\omega)$ certainly exists for real ω, a-fortiori for $\mathrm{Im}(\omega) > 0$, where the integral has an additional real exponential convergence. In fact this implies *analyticity of $g(\omega)$ in the entire upper-half-plane.*

(Note: upper and lower have switched as the modern convention is for our waves to have the time dependence $\propto e^{-i\omega t}$).

But causality requires that there be no scattered wave ahead of the incident one! Thus

$$\psi_{\rm scat}(r,t) = 0, \quad t - \frac{r}{c} < 0$$

so by the same argument, $f(\omega)g(\omega)$ is upper-half-plane analytic. This implies $f(\omega)$ cannot have singularities in the upper-half-plane.

Heisenberg: The Birth of S-Matrix Theory (1942-43) Heisenberg, Sept. 1942 Z. f. Physik 120(1943),513,673:
"The observable quantities in the theory of elementary particles"

The divergence difficulties of quantized field theories led Heisenberg to adopt, as in the great "Umdeutung" work of 1925, an empiricist attitude:

In this situation it seems reasonable to pose the question "what concepts of the present theory can also survive in a future one?", and this question is approximately synonymous with another, namely "which quantities of the present theory are observable". For the future theory should also first and foremost contain relations between "observable quantities". Of course, the question as to which quantities are really observable can only be fully settled once the future theory is in hand. But even before the final theory is attained, the study of difficulties in the earlier one can provide clear indications that certain concepts will have to be abandoned in the future, whereas others are hardly touched by these difficulties…

In this work Heisenberg introduces the concept of a unitary matrix S connecting arbitrary incoming and outgoing multi-particle states.

S	$\vec{k}_1''\vec{k}_2''$	$\vec{k}_1''\vec{k}_2''\vec{k}_3''$	$\vec{k}_1''\vec{k}_2''\vec{k}_3''\vec{k}_4''$
$\vec{k}_1'\vec{k}_2'$	**2->2 elastic scattering**	**2->3 inelastic ("Emission")**	**2->4 processes**
$\vec{k}_1'\vec{k}_2'\vec{k}_3'$	**3->2**	**3->3**	**3->4**
$\vec{k}_1'\vec{k}_2'\vec{k}_3'\vec{k}_4'$	**4->2**	**4->3**	**4->4**

outgoing state

incoming state

(Heisenberg (12a))

The implications of unitarity of S are carefully discussed in this work. In particular,

1. The possibility of writing any unitary matrix $S = e^{i\eta}$, where η is *Hermitian* suggests abandoning the Hermitian Hamiltonian operator H in favor of η as specifying the basic physical content of the theory.

2. From the unitarity condition $S^\dagger S = 1$, and writing $S = 1 + R$, where R represents the nontrivial scattering in the theory, Heisenberg derives the nonlinear relation

$$R^\dagger R = -(R + R^\dagger)$$

> *Here we clearly encounter the fact..., that the connections between the scattering coefficients contain linear and quadratic parts...*

In October 1943, Heisenberg presented his results on S-matrix theory in an "informal" colloquium in Leiden. At the colloquium (M. Dresden, "H.A. Kramers, Between Tradition and Revolution", p.454):

> Kramers made the penetrating remark that to have any hope that the S matrix would be determined by general physical principles (and not by particular forces and potentials), it was essential to consider the S matrix as an analytic function of the complex momentum variables.

The importance of this idea was explicitly acknowledged by Heisenberg in a letter to Kramers a few weeks later:

> Since my return I have thought extensively about your idea to consider the η-matrix as an analytic function. I am more and more thrilled by your proposal, because I believe that in this way one can really obtain a complete theory of elementary particles..

Kramers (Address "Fundamental Difficulties of a theory of particles", given at the Symposium on Elementary Particles, Utrecht 14/4/1944):

> Heisenberg's recent investigations concerning the possibility of a relativistic description of the interaction that is not based on the use of a Hamiltonian.. Heisenberg considers only free particles and introduces a formalism ('scattering matrix') by means of which the result of a short interaction between these particles can be described. ..It is interesting that the scattering matrix is also able in principle to answer the question, in which stationary states the particles considered can be bound together.These are related to the existence and the position of zeros and poles of the eigenvalues of the scattering matrix, *considered as a complex function of its arguments.*

What Kramers knew (and Heisenberg didn't) in 1943

The collaboration with Wouthuysen shows that in 1943 (at the time of Heisenberg's visit) Kramers clearly understood:

(A) The existence of bound states in a theory *is visible in the analytic continuation of the scattering amplitudes of the theory.* In a sense, knowledge of the ionized states implies knowledge of the bound ones. This is just completeness: $1 = \sum_n |n><n| + \int dE|E><E|$.

Example: low energy electron-proton scattering amplitude

$$f(E) \propto \Gamma(1 - i\frac{\sqrt{m}e^2}{\hbar\sqrt{2E}})$$

which has simple poles at

$$1 - i\frac{\sqrt{m}e^2}{\hbar\sqrt{2E}} = 1 - n \quad , \quad n = 1, 2, 3, ...$$

i.e. when

$$E = -\frac{me^4}{2\hbar^2 n^2}$$

In addition, Kramers almost surely understood (B) the connection between unitarity and cut singularities (imaginary parts for real energy) of amplitudes. Heisenberg *almost* has this! Instead of writing $S = 1 + R$, let's separate the nontrivial scattering matrix as $S = 1 + iT$, where now

$$S^\dagger S = 1 \Rightarrow T^\dagger T = i(T^\dagger - T)$$

Take the matrix element of the T-matrix unitarity condition between a specific initial state i and final state f:

$$\sum_n T^*_{nf} T_{ni} = i(T^*_{if} - T_{fi})$$

For forward scattering, $f = i$,

$$2\mathrm{Im}(T_{ii}) = \sum_n |T_{ni}|^2$$

This is the famous "optical theorem". Intuitively, it simply says that particles scattered *out* of the forward direction (absorption, in optical terms) appear in the other directions (ie the states n).

From the optical theorem

$$2\mathrm{Im}(T_{ii}) = \sum_n |T_{ni}|^2$$

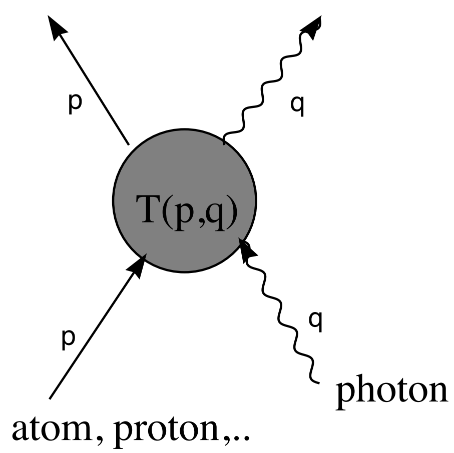
we learn immediately two critical facts:

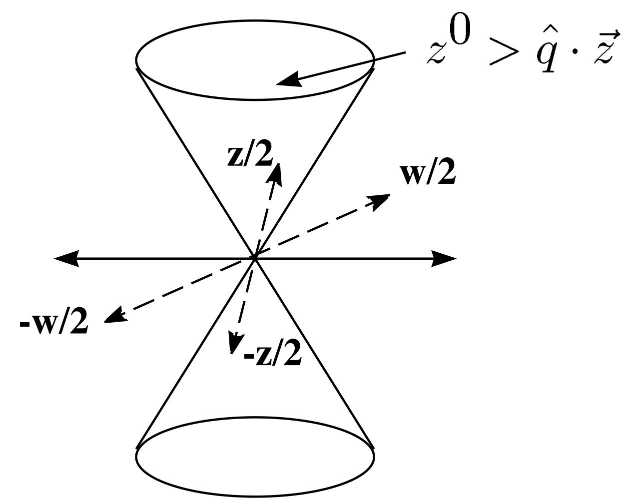
1. If we analytically continue the forward scattering amplitude T_{ii} to a (real) energy where there are no available physical states n, $\mathrm{Im}(T_{ii}) = 0$. So we are dealing with analytic functions which, at least somewhere on the real axis, are real: they are "real-analytic functions". The Schwarz reflection principle then says that the value of such functions in the lower half-plane is the complex conjugate of the value in the upper half-plane. Hence, for such amplitudes, *upper half-plane analyticity* $\Rightarrow$ *lower half-plane analyticity* $\Rightarrow$ *analyticity everywhere except for cuts and (possibly) bound-state poles on the real axis.*

2. The discontinuity $\mathrm{Im}(T_{ii})$ of T_{ii} across cuts switches on once physical states n appear with the energy and momentum of the incoming state i. In other words, unitarity requires the existence of cuts of the scattering amplitude on the real axis. The *principle of maximal analyticity* states that these are the *only* cut singularities of the scattering amplitude. *Never proved*, it played during the 1960s a central - indeed indispensable- role in S-matrix theory.

Quantum Field Theory and Dispersion Relations
(Ref: M. Gell-Mann, M.L. Goldberger and W. Thirring, PR **95** (1954) 1612.)

Let $|p>$ be a particle (atom, proton,..) of four-momentum p, off which we scatter a photon of four-momentum $q = (\omega, \omega\hat{q})$. If the scattering is in the forward direction, the final state atom and photon have the same momenta p, q after, and the scattering amplitude is a function $T(p, q)$ of p and q only.





$[j(\frac{z}{2}), j(-\frac{z}{2})] \neq 0$ causally connected

$[j(\frac{w}{2}), j(-\frac{w}{2})] = 0$ causally independent (microcausality condition)

The methods of quantum field theory developed in the early 50's allow us to write $T(p, q)$ in terms of the Fourier transform of a *retarded commutator* of the four-vector electromagnetic charge-current operator $j_\mu(x)$, as follows:

$$T(p, q) \propto \int d^4 z e^{iq\cdot z} \theta(z^0) < p|[j_\mu(\frac{z}{2}), j^\mu(-\frac{z}{2})]|p >$$

The only interesting dependence of the forward scattering amplitude $T(p,q)$ is on the photon frequency ω, so we normally denote it simply $f(\omega)$:

$$\begin{aligned} f(\omega) &\propto \int d^4z e^{iq\cdot z}\theta(z^0) < p|[j_\mu(\frac{z}{2}), j^\mu(-\frac{z}{2})]|p > \\ &\propto \int_0^{+\infty} dz^0 \int d\vec{z} e^{i\omega(z^0 - \hat{q}\cdot\vec{z})} < p|[j_\mu(\frac{z}{2}), j^\mu(-\frac{z}{2})]|p > \end{aligned}$$

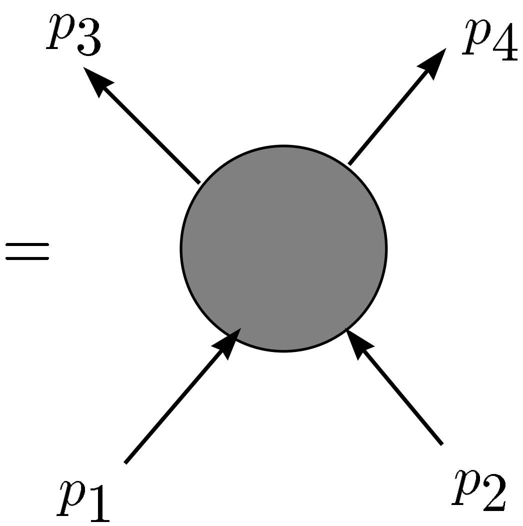
Now write $\omega = \omega_R + i\omega_I$ and note that because the commutator is only nonzero when $z^0 - \hat{q}\cdot\vec{z} > 0$ (forward light-cone), existence of the integral for $\omega_I = 0$ implies even better convergence for $\omega_I > 0$, i.e. in the upper-half-plane. Well known theorems of complex analysis imply full upper-half-plane analyticity of $f(\omega)$. Repeating the Kramers arguments, we again have

$$\mathrm{Re} f(\omega) = \frac{2}{\pi} P \int d\omega' \frac{\omega' \mathrm{Im} f(\omega')}{\omega'^2 - \omega^2}$$

(Comment and confession: subtractions needed!)

The 2→2 Scattering Amplitude in Relativistic Field Theory

$A(s,t,u) =$



$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

$$p_1 + p_2 = p_3 + p_4 \;\Rightarrow\; s + t + u = 4m^2$$

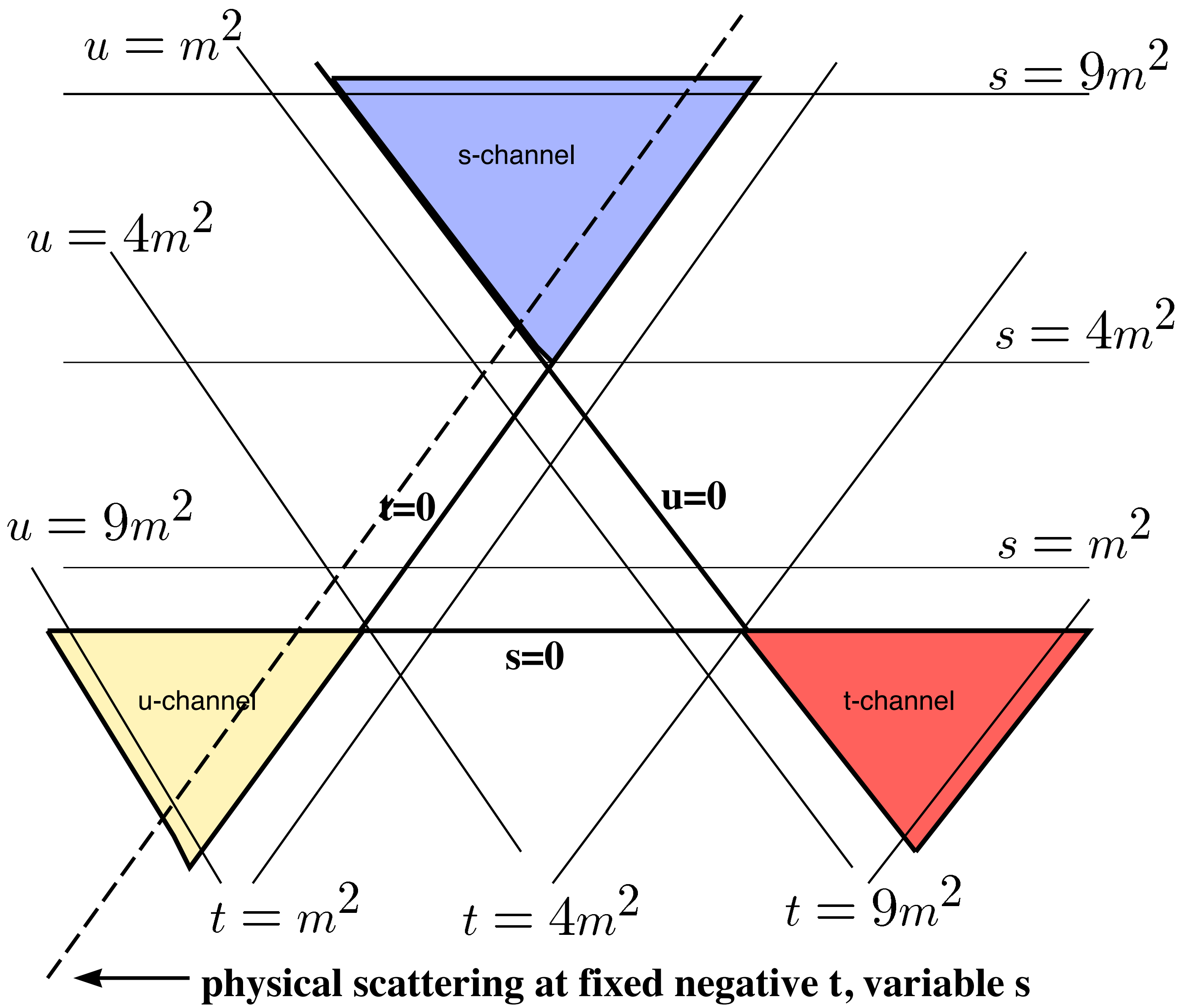
$\Rightarrow$ **the scattering amplitude can be written** $A(s,t)$.

A deep consequence of relativity and locality is the *crossing symmetry* of scattering amplitudes, which says that incoming/outgoing particles can be replaced by outgoing/incoming antiparticles with the four-momenta reversed. In other words, the amplitudes for the processes

$$\begin{aligned} A(p_1) + B(p_2) &\rightarrow C(p_3) + D(p_4) \\ A(p_1) + \bar{C}(-p_3) &\rightarrow \bar{B}(-p_2) + D(p_4) \end{aligned}$$

are identical- corresponding to an interchange of the s *and* t *variables.* Of course, *for physical scattering* $s > 0, t < 0$, so the t-channel amplitude $A(s < 0, t > 0, u)$ for the crossed process $A + \bar{C} \rightarrow \bar{B} + D$ requires analytic continuation from the physically accessible region $A(s > 0, t < 0, u)$ for the s-channel process $A + B \rightarrow C + D$. Once again, *analyticity* is central!

The Mandelstam Diagram: Physical Regions for 2-2 scattering



For scattering of equal mass m particles (e.g. pion-pion scattering), unitarity requires that imaginary parts of $A(s,t)$ appear at the lowest value of total energy-momentum squared for any new set of states that can be produced by the incoming (two) particles. Thus, there is a two-particle cut that begins at $s = (2m)^2 = 4m^2$, a three-particle cut beginning at $s = 9m^2$, etc. These cuts, starting at positive values of s, are called right-hand cuts. For fixed t, the u-channel cuts starting at $u = 4m^2$ imply also *left-hand cuts* in the s variable running to the left.

Regge Behavior

For equal mass elastic scattering, $s = 4(k^2 + m^2)$, $t = 2k^2(\cos(\theta) - 1)$ and a partial wave expansion for the scattering amplitude reads

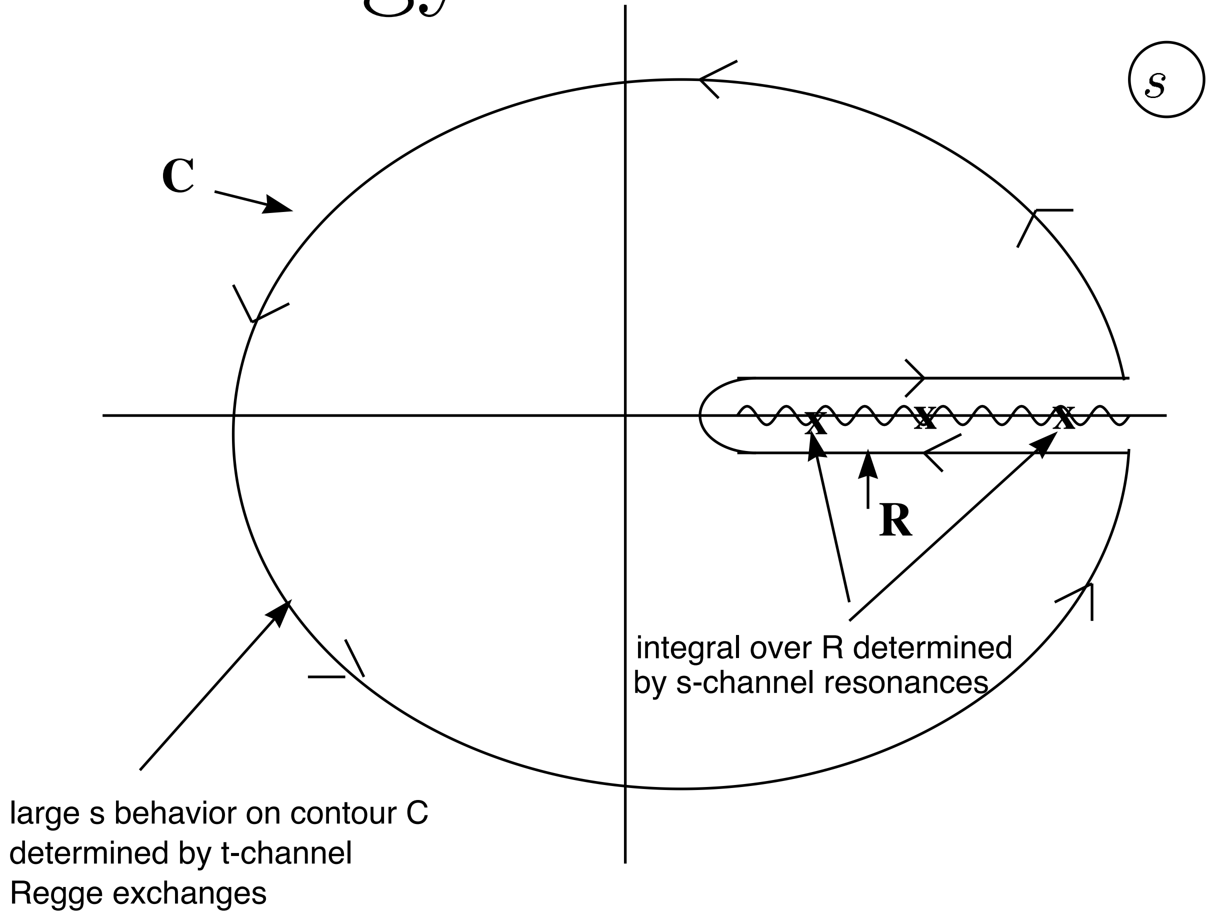
$$A(s,t) \propto \sum_l f_l(s) P_l(\cos(\theta)) = \sum_l f_l(s) P_l(\cos(\theta))$$

$$\rightarrow t^{\alpha_n(s)}, \quad \text{fixed } s, \ t \rightarrow \infty$$

where the partial wave amplitude $f_l(s)$ has poles when analytically continued to complex l at $l = \alpha_n(s)$ (the so-called "Regge poles"). By crossing, this means that the behavior of $A(s,t)$ at fixed t, large s is determined by the *exchange* of resonances of squared-mass t and spin $\alpha_n(t)$. The dependence of $\alpha_n(t)$ was found empirically in the 1960's to be astonishingly simple:

$$\alpha_n(t) \simeq \alpha_0 + \alpha' t$$

Finite Energy Sum Rules and Duality



Equating the integral along the real axis cut (approximately given by a sum of s-channel resonances) to the contribution of the circular contour at large s (approximately given by a sum of exchanged t-channel Regge poles) one arrives at the concept of **Global Duality**:

the average of s-channel resonances $\simeq$ *sum of t-channel exchanges*

The much stronger concept of **Local (or "Exact") Duality** soon followed: the scattering amplitude can be written exactly as *either* a sum of s-channel resonances created *by* the incoming particles, *or* as a sum of t-channel resonances exchanged *between* the incoming particles

$$A(s,t) \quad = \quad \sum_n \frac{g_n(t)}{s - M_n^2} \quad = A'(s,t) = \sum_n \frac{g'_n(s)}{t - M_n'^2}$$

A Toy Model with exact duality: the Veneziano Model (1968)

A model in which the scattering amplitude can be expressed exactly as *either* a sum over direct (s) channel resonances *or* exchanged (t) channel resonances was found by Veneziano. He proposed

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}, \quad \alpha(s) = \alpha_0 + \alpha' s$$

This is just Euler's beta function, which can be rewritten exactly in the form $A(s,t) = \sum_n \frac{g_n(t)}{s-M_n^2}$, where the resonance squared masses are linearly related to the spin n of the resonance ("linear Regge trajectories"):

$$\begin{aligned} M_n^2 &= -\frac{\alpha_0}{\alpha'} + \frac{1}{\alpha'} n \\ g_n(t) &= -\frac{1}{\alpha'} \frac{(\alpha(t)+1)(\alpha(t)+2)..(\alpha(t)+n)}{n!} \end{aligned}$$

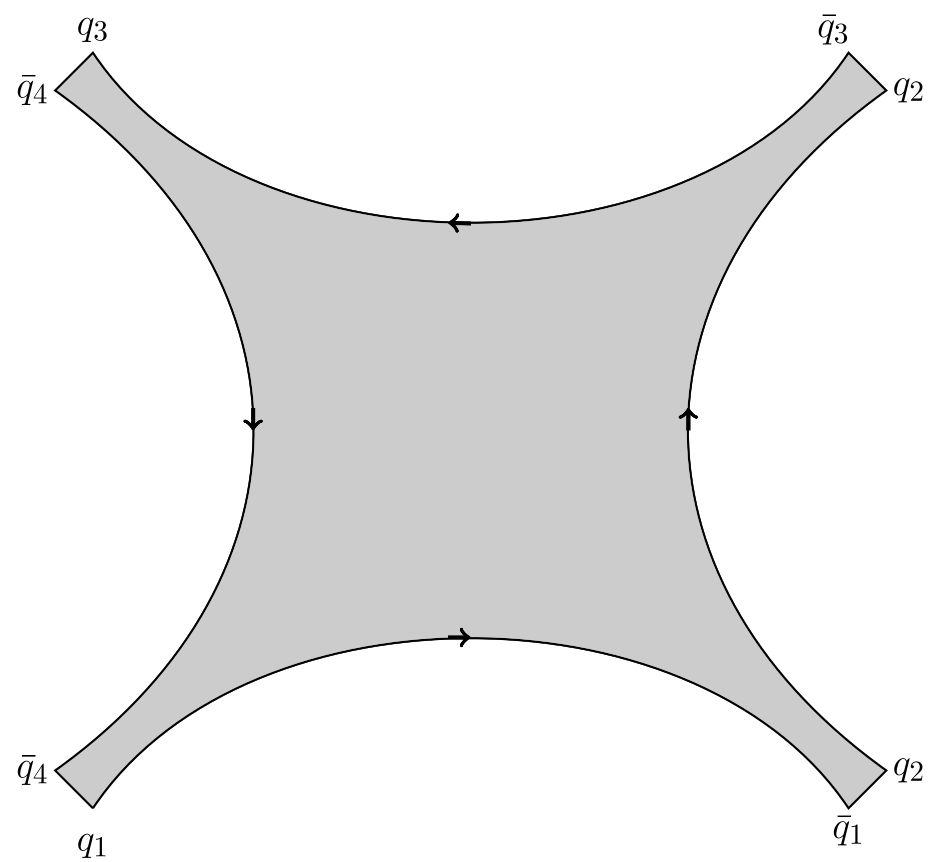
But by construction, $A(s,t) = A(t,s)$*!*, so we can equally well write the scattering amplitude as a sum of exchanged t-channel resonances

$$A(s,t) = \sum_n \frac{g_n(s)}{t-M_n^2}$$

This is completely foreign to conventional local field theory, in which the complete amplitude involves *separate* s-channel and t-channel resonances, with no a-priori connection between them!

A Microphysics for Dual Resonance Models: String Theory (Nambu,Nielsen 1969-1970)

View mesons as bound states of quarks and antiquarks, but *connected by a one dimensional fluctuating string*: in this picture $\pi^a(= q_1\bar{q_4})$-$\pi^b(= q_2\bar{q_1})$ scattering looks like



In string theory, there is only a single (dual) amplitude, and $A(s,t) = A'(s,t)$.

Final Status of the Analyticity Principle

Although in some very limited cases the required analyticity for S-matrix dispersion theory could be established rigorously on the basis of general field-theoretic principles, in almost all cases, the use of analyticity was a "seat of the pants" affair, justified by vague philosophical mumblings-

A philosophical objection may be raised against the S matrix approach, that is, that the principle of analyticity has no physical basis, whereas in field theory it appears related to the notion of microscopic causality. My personal inclination here is to resurrect the ancient principle of "lack of sufficient reason". I assert that it is natural for an S matrix element to vary smoothly as energies and angles are changed, and that a natural mathematical definition of physical smoothness lies in the concept of analyticity. The fundamental principle therefore might be one of maximum smoothness: the S matrix has no singularities except where absolutely necessary to satisfy unitarity. There is no "reason" for it to have any others.(Geoffrey Chew, 1961)

The most expansive, and rigorous, formulation of the analyticity principle is to be found in the Hall-Wightman theorem of axiomatic quantum field theory, which establishes a very large region of analyticity for the n-point Wightman functions of the theory, continued to complexified Minkowski space, on the basis of spectral and microcausality axioms. Unfortunately, this theorem has very little to say directly about the analyticity properties in momentum variables of the on-mass-shell limits of these functions relevant to S-matrix theory.

Summary

- The direct empirical character of the amplitudes described in quantum dispersion theory (in the early 1920s, the forward scattering amplitudes of photons on atomic or molecular bound states) encouraged the development of matrix mechanics as a dynamical formalism relating only directly observable quantities. One thereby circumvented the conceptual difficulties engendered by an attachment to classical phase space and Bohr orbits. In the end, one is led to a fully consistent microtheory (modern quantum theory) which goes well beyond its empiricist origins.

- In later developments, after the introduction of the concept of the S-matrix (Heisenberg), and exploitation of the powerful implications of analyticity ideas, dispersion theory reprises its role as midwife. Here, the conceptual difficulties to be circumvented were the ultraviolet infinities initially encountered in the development of quantum field theory (Heisenberg, 1940s), and later (1960s) the resistance of the strong interactions to interpretation in terms of a conventional renormalizable Lagrangian field theory. In this case, the exploitation of dispersion theory was to lead to a new symmetry of amplitudes, duality, and eventually to a new microtheory of elementary particles: string theory. The ultimate consistency and empirical adequacy of this theory is as yet unresolved.