Diagrammatic Quantum (Quasi-)Monte Carlo
Out of equilibrium / in real time

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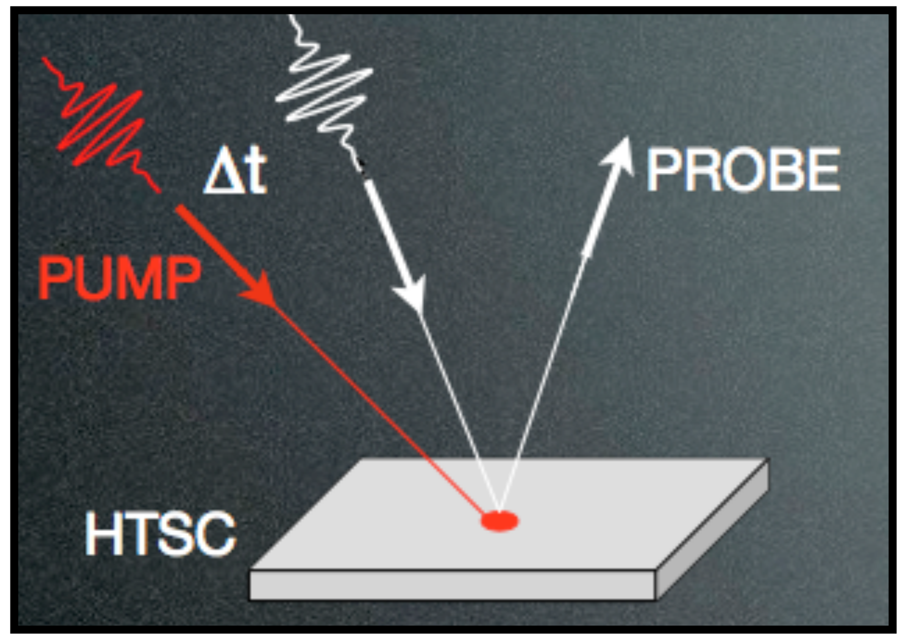
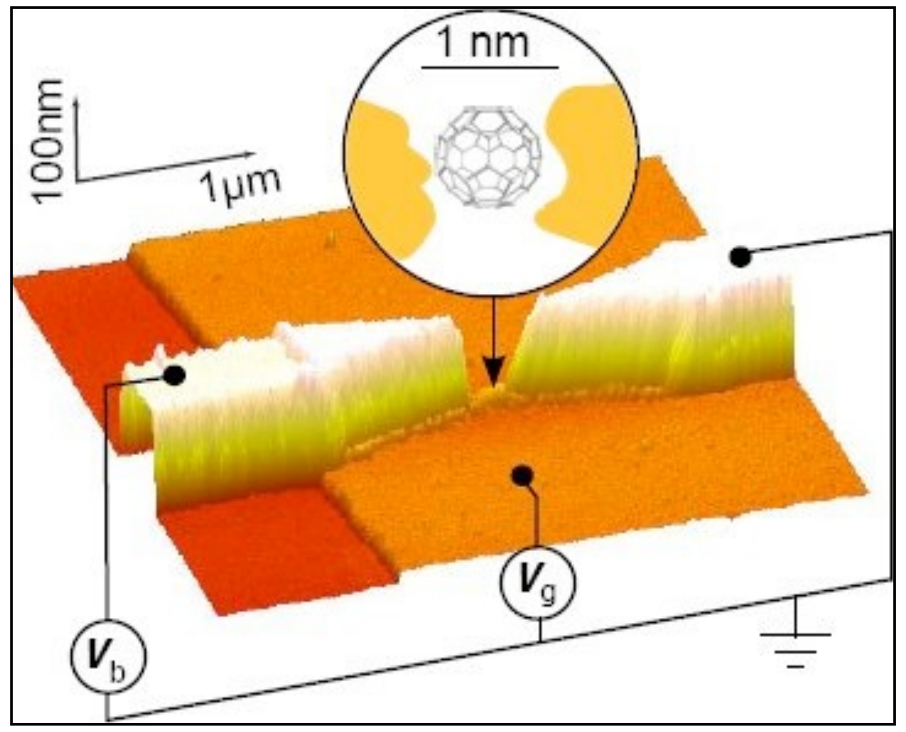
Out of equilibrium & strong correlations

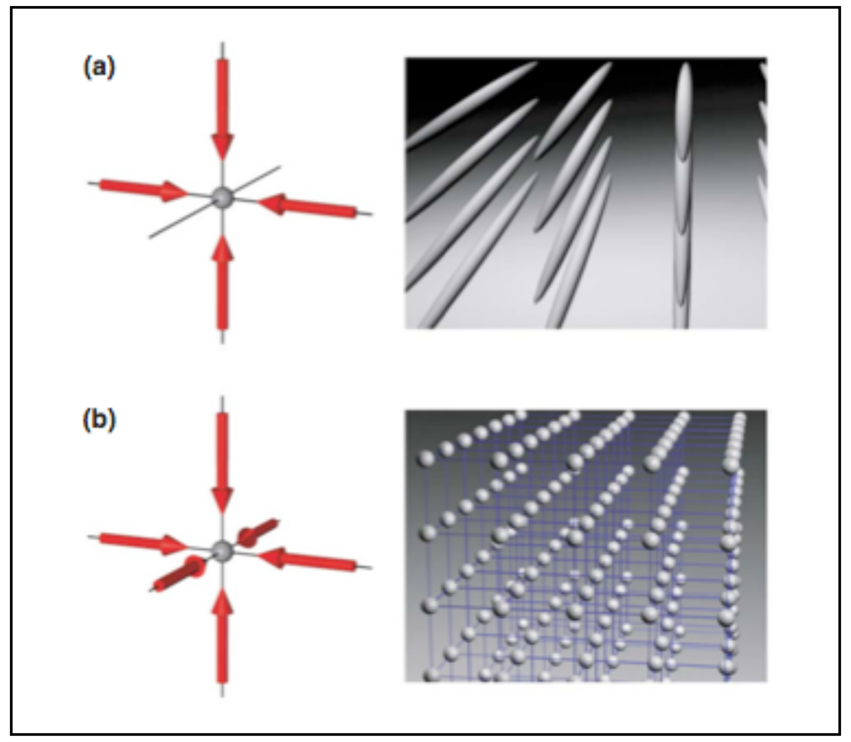
- Many experiments : Pump probe, quantum dots, ultra-cold atoms, cavities.



Pump probe



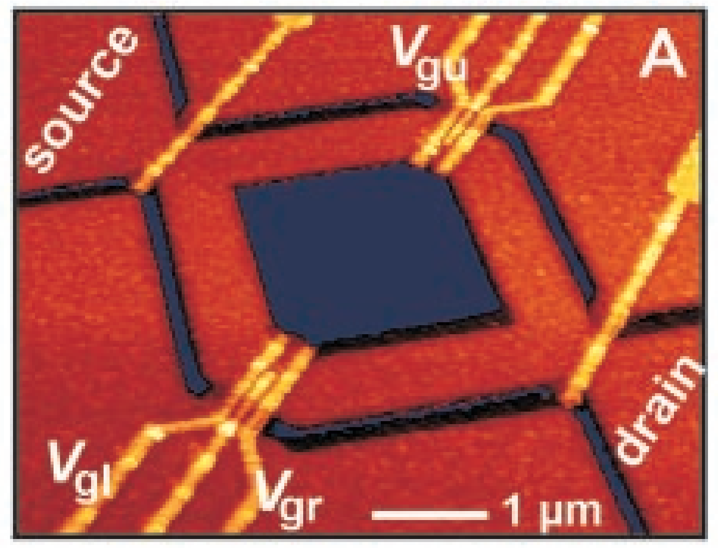
Nano-electronics



Ultra-cold atoms

- Computational physics challenge :
  - Exact methods for out of equilibrium systems, at strong coupling
  - Control, speed and precision
  - Long time (after quench), steady state. Resolve various energy/time scales. Early Monte Carlo have sign problem *Muelbacher et al. PRB 2009;Werner et al PRB 2009; Schiro PRB 2009.*

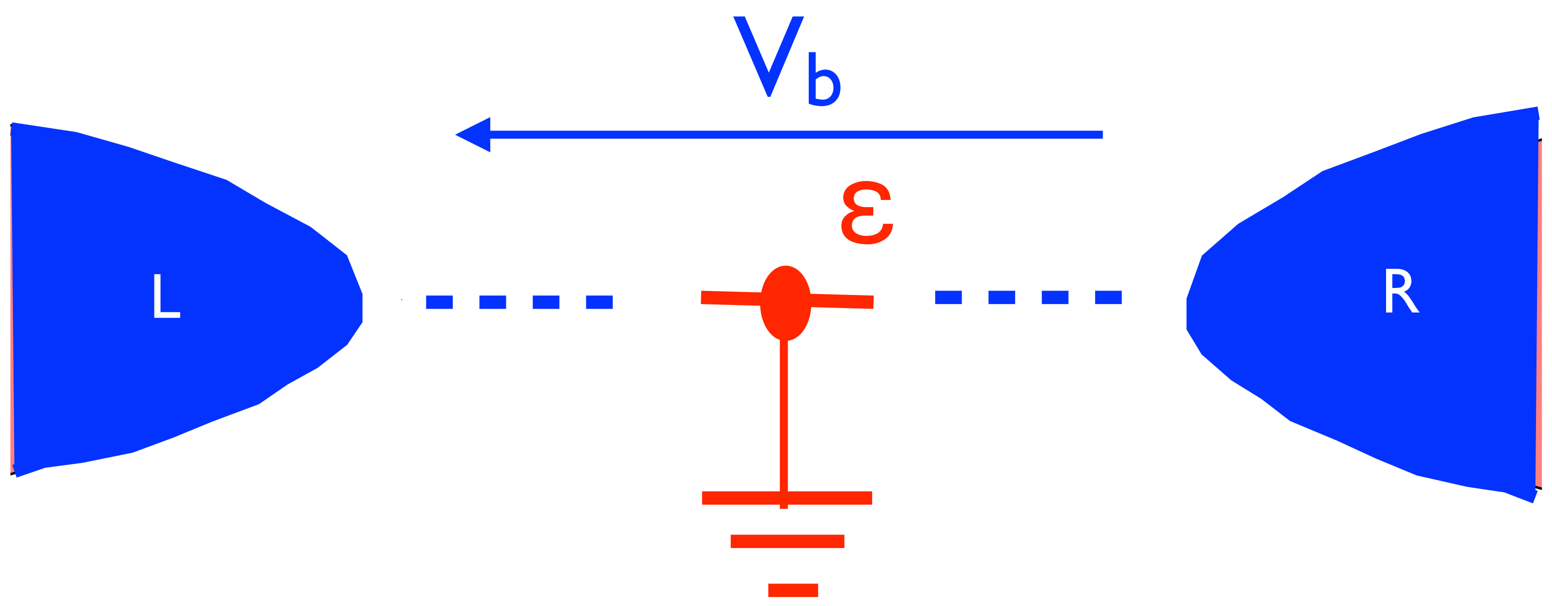
Example : a simple model for a quantum dot



- Anderson model with two leads (L, R).

Bath *Local orbital* *Hybridization*

$$H = \sum_{\substack{k\sigma \\ \alpha = L,R}} \varepsilon_{k\alpha} c^\dagger_{k\sigma\alpha} c_{k\sigma\alpha} + \sum_{\sigma} \varepsilon_d d^\dagger_\sigma d_\sigma + U n_{d\uparrow} n_{d\downarrow} + \sum_{\substack{k\sigma \\ \alpha = L,R}} g_{k\sigma\alpha} (c^\dagger_{k\sigma\alpha} d_\sigma + h.c.)$$



Level width at U=0

$$\Gamma = \pi \rho_{E_F} g^2$$

- Questions : Spectral function ? Kondo temperature ? Current ?

We want a precise solution, at low temperature, any V_b, in steady state

Summary of the approach

- Perturbation theory in interaction U (10-15 orders) for physical quantities.

$$Q(t, U) = \sum_{n=0}^{K} Q_n(t) U^n$$

Time

Interaction

1. Works even at long time, even in strong coupling regime (e.g. Kondo effect)

2. How to compute $Q_n(t)$? Cost $O(2^n)$.
   High dimensional integrals.
   Real time "diagrammatic" Quantum Monte Carlo
   Beyond stochastic methods : Quasi-Monte Carlo (QQMC)

3. How to sum the series ?

- See also : Expansion around atomic limit.
  "Inchworm" approach. *Cohen, Gull, Reichman, Millis PRL (2015)*

Schwinger-Keldysh
1- Notations

Three diagrammatic techniques.

- T=0 : Ground state
- Matsubara : finite T, in thermal equilibrium
- Schwinger-Keldysh
  - General. Equilibrium or out of equilibrium. Real time.
  - A bit more complex technically.
    It is not possible to write diagrams with only one Green function.
  - Conceptually simpler.
    Bath are explicitly included, no hidden relaxation (or Gell-Man Low theorem).

Notations

- Canonical fermion operator
- a,b = multi-index : k, x, spin, ... everything but time.

$$\{c_a, c_b^\dagger\} = \delta_{ab}$$

- Chronological product

$$TA(t)B(t') = \theta(t-t')A(t)B(t') + \zeta_{AB}\theta(t'-t)B(t')A(t)$$
$$\check{T}A(t)B(t') = \theta(t'-t)A(t)B(t') + \zeta_{AB}\theta(t-t')B(t')A(t)$$

$\zeta_{AB} = \pm 1$ A, B both fermionic ? -1 else +1

Hamiltonian evolution

- Total Hamiltonian of the system, e.g.

$$H = H_{dot} + H_{bath} + H_{dot\text{-}bath}$$

- H(t) determines the dynamics in real time. Can be time dependent.

- Evolution operator U_H : evolves the state of the system from t_0 to t

$$|\psi(t)\rangle = U_H(t,t_0)\,|\psi(t_0)\rangle$$

- Heisenberg representation for operator A

$$A(t) \equiv U_H^\dagger(t,t_0)A(t_0)U_H(t,t_0)$$

Reminder : density matrix

- For the whole system (e.g. dot + baths)
- Describes the occupation of the levels.

$$
\begin{aligned}
\mathrm{Tr}\rho &= 1\\
\rho^\dagger &= \rho\\
\rho &\geq 0\\
i\partial_t \rho(t) &= [H(t), \rho(t)]\\
\rho(t) &= U_H(t,t_0)\rho(t_0)U_H^\dagger(t,t_0)\\
\langle A(t)\rangle &\equiv \mathrm{Tr}\big(\rho(t)A(t_0)\big) = \mathrm{Tr}\big(\rho(t_0)A(t)\big) \qquad \text{for any operator } A
\end{aligned}
$$

Study evolution of ρ
or correlators

- Out of equilibrium : 2 independents objects. H and ρ.

- Thermal equilibrium :

$$\bar\rho = \frac{1}{Z}e^{-\beta H}, \qquad Z = Tre^{-\beta H}$$

$$\bar\rho = \frac{1}{Z}e^{-\beta(H-\mu\hat N)}, \qquad Z = Tre^{-\beta(H-\mu\hat N)}$$

One particle Green functions

- Definitions

$$G^{++}_{ab}(t,t') \equiv -i \left\langle T c_a(t) c^\dagger_b(t') \right\rangle$$

+,- : just notations
for the moment

$$G^{--}_{ab}(t,t') \equiv -i \left\langle \check{T} c_a(t) c^\dagger_b(t') \right\rangle$$

$$G^{+-}_{ab}(t,t') = G^{<}_{ab}(t,t') \equiv i \left\langle c^\dagger_b(t') c_a(t) \right\rangle$$

$$G^{-+}_{ab}(t,t') = G^{>}_{ab}(t,t') \equiv -i \left\langle c_a(t) c^\dagger_b(t') \right\rangle$$

- Only 2 Green functions are independents (from the definition of T)

$$G^{++}_{ab}(t,t') = \theta(t-t') G^{>}_{ab}(t,t') + \theta(t'-t) G^{<}_{ab}(t,t')$$
$$G^{--}_{ab}(t,t') = \theta(t'-t) G^{>}_{ab}(t,t') + \theta(t-t') G^{<}_{ab}(t,t')$$

- In equilibrium, only one !
  Fluctuation-Dissipation theorem, Kubo-Martin-Schwinger relation

$$\left\langle c^\dagger_b(t') c_a(t) \right\rangle = \left\langle c_a(t) c^\dagger_b(t'+i\beta) \right\rangle \qquad G^{<}_{ab}(\omega) = -e^{-\beta\omega} G^{>}_{ab}(\omega)$$

Schwinger-Keldysh
2- Diagrammatic expansion

General strategy

- Start at t = t_0 (=0 in most slides below)
- With initial condition :
  $\rho = \rho_0$ at thermal equilibrium with <u>non interacting</u> Hamiltonian H_0
  at a temperature β
- NB : it is possible to start with interacting equilibrium. Baym-Kadanoff contour.
  Not covered here.
- Study the expansion of correlators at finite time.

$$\mathrm{Tr}(\rho_0 A(t) B(t')...)$$

- Build the diagrammatic at finite time.
- If needed, take the limit $t, t' \to \infty$ or $t_0 \to -\infty$
- Separate diagrams technique & thermalization/relaxation/bath questions.

Interaction picture

- Hamiltonian evolution of whole system (dot + bath)

$$H = H_0 + V(t)$$

Non interacting part

Interaction (U term)

- Operator in interaction picture (≠ Heisenberg picture).

$$\hat{A}(t) \equiv e^{iH_0 t} A e^{-iH_0 t}$$

- Evolution operator in interaction picture

$$U(t) \equiv e^{iH_0 t} U_H(t)$$

$$\begin{aligned} i\partial_t U(t) &= \hat{V}(t) U(t) \\ U(0) &= 1 \end{aligned} \quad \longrightarrow \quad U(t) = T \exp\left(-i \int_0^t \hat{V}(u) du \right)$$

Time evolution of a physical quantity

- Start at t=0 (t0) from a non-interacting equilibrium density matrix ρ_0

$$U(t) = T \exp\left(-i \int_0^t \hat{V}(u) du\right)$$

- Average of an operator A

Average in initial state

$$\begin{aligned}\langle A(t)\rangle &= \mathrm{Tr}\left(\rho_0 A(t)\right)\\ &= \mathrm{Tr}\left(\rho_0 \left(U(t)\right)^\dagger \hat{A}(t) U(t)\right)\\ &= \mathrm{Tr}\left(\rho_0 \check{T} \exp\left(+i \int_0^t \hat{V}(u) du\right) \hat{A}(t) T \exp\left(-i \int_0^t \hat{V}(u) du\right)\right)\end{aligned}$$

- Expand the *exp*.

- Problem : not a T ordered product ! How to use a Wick theorem ?

Wick theorem : reminder

- H_0 a quadratic (gaussian) Hamiltonian for fermions

$$H_0 = c_a^\dagger M_{ab} c_b$$

- Then the N body correlator is given by (ζ(P) is the signature of P)

$$\left\langle T c_{a_1}(t_1) \dots c_{a_n}(t_n) c^\dagger_{a'_n}(t'_n) \dots c^\dagger_{a'_1}(t'_1) \right\rangle_0 = \sum_{P \in S_n} \zeta(P) \prod_{k=1}^{n} \left\langle T c_{a_k}(t_k) c^\dagger_{a'_{P(k)}}(t'_{P(k)}) \right\rangle_0$$

$$= \det_{1 \le i,j \le n} \left[\left\langle T c_{a_i}(t_i) c^\dagger_{a'_j}(t'_j) \right\rangle_0 \right]$$

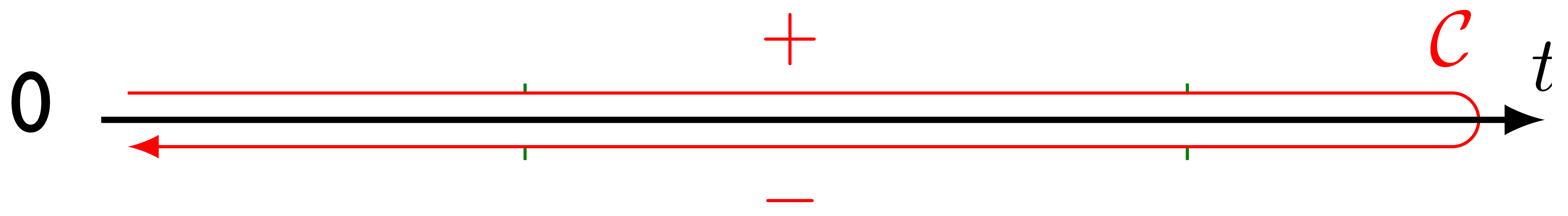
$$\langle X \rangle_0 \equiv \frac{1}{Z_0} Tr \left(e^{-\beta H_0} X \right)$$

$$Z_0 = Tr \left(e^{-\beta H_0} \right)$$

- Requires a “gaussian” density matrix ρ_0
- Wick theorem is valid on any contour, as long as a time ordering is defined.

Schwinger Keldysh double contour

- Every times is now a couple (t,a), a = ± 1 (Keldysh indices)



$$\langle A(t)\rangle = \mathrm{Tr}\left(\rho_0 \check{T}\exp\left(+i\int_0^t \hat{V}(u)du\right)\hat{A}(t)T\exp\left(-i\int_0^t \hat{V}(u)du\right)\right)$$

$$= \left\langle T_{\mathcal{C}}\hat{A}(t)\ \exp\left(-i\int_{\mathcal{C}} \hat{V}(u)du\right)\right\rangle$$

- Correlation function

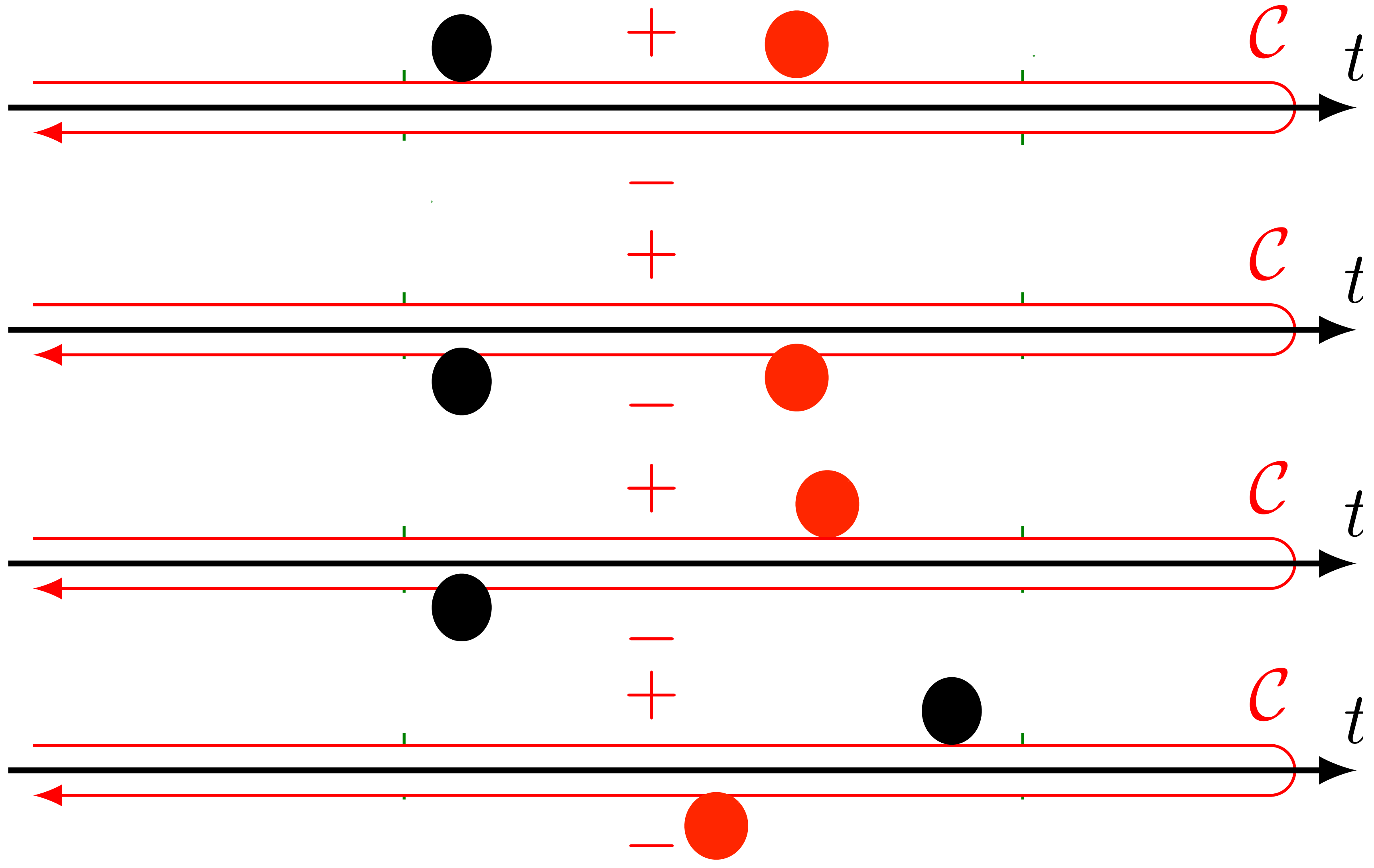
$$\langle T_{\mathcal{C}}A(t,\alpha)B(t',\alpha')\rangle = \left\langle T_{\mathcal{C}}\hat{A}(t,\alpha)\hat{B}(t',\alpha')\ \exp\left(-i\int_{\mathcal{C}} \hat{V}(u)du\right)\right\rangle$$

- Diagrams : expand the exponential.

Fundamental relation

- Connect the notations +/- to the double contour

$$\mathbf{G} \equiv -i \left\langle T_{\mathcal{C}} \underset{\bullet}{c_a(t,\alpha)} \underset{\bullet}{c_b^\dagger(t',\alpha')} \right\rangle = \begin{pmatrix} G_{ab}^{++}(t,t') & G_{ab}^{+-}(t,t') \\ G_{ab}^{-+}(t,t') & G_{ab}^{--}(t,t') \end{pmatrix}$$



$$G_{ab}^{++}(t,t') \equiv -i \left\langle T c_a(t) c_b^\dagger(t') \right\rangle$$

$$G_{ab}^{--}(t,t') \equiv -i \left\langle \check{T} c_a(t) c_b^\dagger(t') \right\rangle$$

$$G_{ab}^{+-}(t,t') \equiv i \left\langle c_b^\dagger(t') c_a(t) \right\rangle$$

$$G_{ab}^{-+}(t,t') \equiv -i \left\langle c_a(t) c_b^\dagger(t') \right\rangle$$

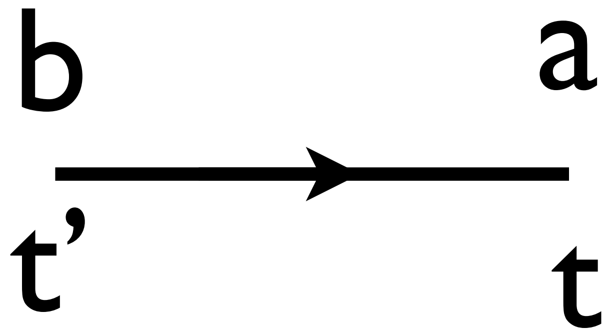
Diagrammatics

- Same diagrams (topology, ...) as ordinary T=0 (or Matsubara) diagrams.
  But with an additional index α for each time

- Any diagrammatic approximation (large N, DMFT, ….) can be generalized to non equilibrium

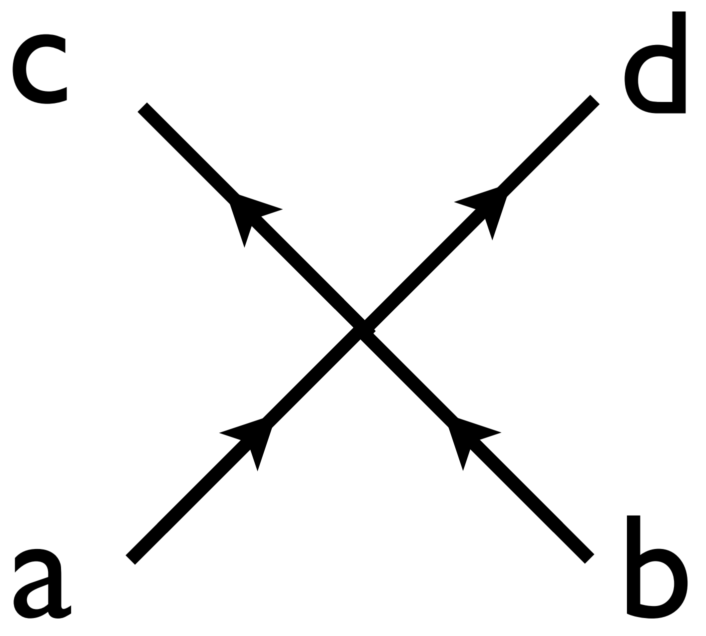
T=0 “ordinary formalism”

Propagator



$iG_{ab}(t,t')$

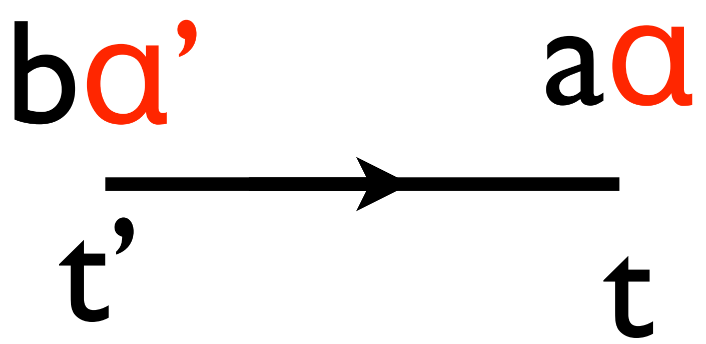
Vertex



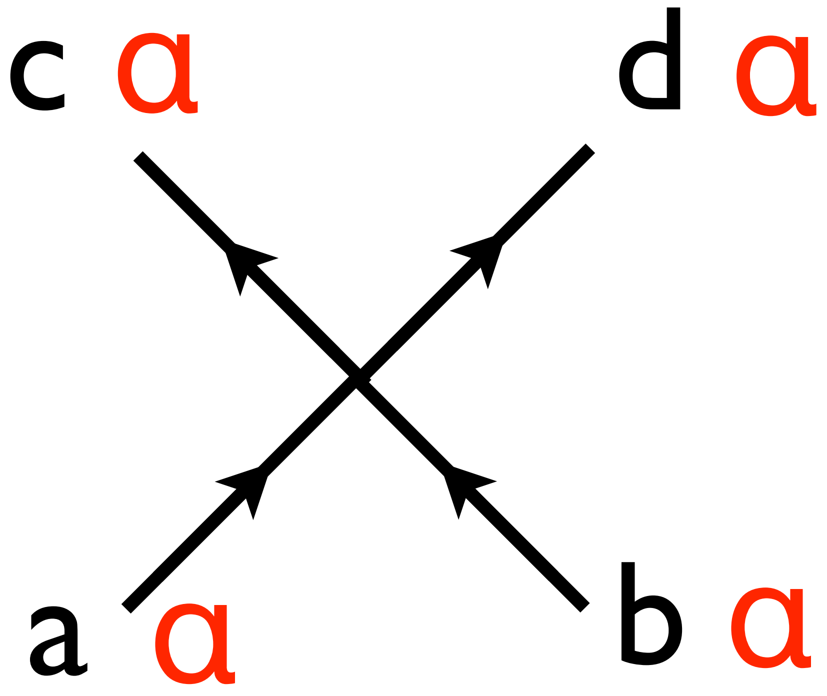
V_{abcd}

- Vacuum diagrams canceled by denominator

Keldysh



$iG_{ab}^{\alpha\alpha'}(t,t')$



αV_{abcd}

- No Vacuum diagram
  Z=1

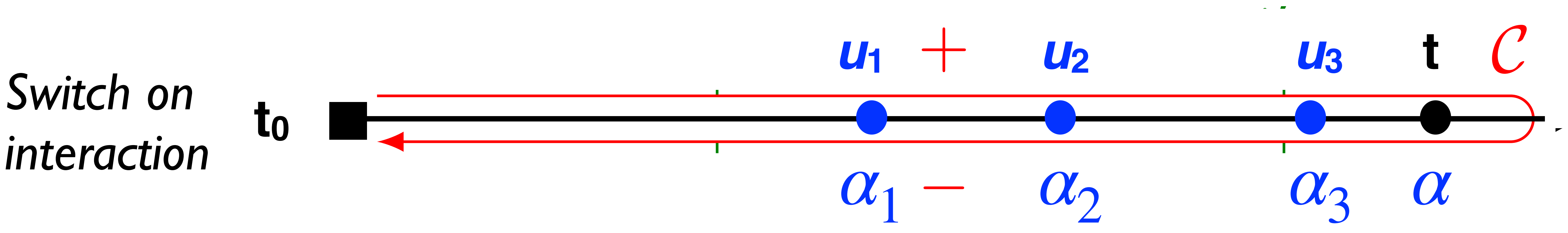
Z=1

$$\begin{aligned}\langle A(t)\rangle &= \mathrm{Tr}\left(\rho_0 A(t)\right)\\ &= \mathrm{Tr}\left(\rho_0 \left(U(t)\right)^\dagger \hat{A}(t) U(t)\right)\\ &= \mathrm{Tr}\left(\rho_0 \check{T}\exp\left(+i\int_0^t \hat{V}(u)du\right)\hat{A}(t) T\exp\left(-i\int_0^t \hat{V}(u)du\right)\right)\end{aligned}$$

- A = 1 . <1> = 1
- No “partition function”, no “vacuum diagrams”

How to compute $Q_n(t)$?

- Schwinger-Keldysh formalism
  Q_n is a n-dimensional integral



Vertices. Times u_i.
Keldysh indices α = -1, 1

$$Q_n(t) = \frac{1}{n!}\boxed{\int_{t_0}^{\infty} du_1 \ldots du_n}\ \boxed{\left(\sum_{\alpha_i = \pm 1} \prod_i \alpha_i \ \det(\ldots) \right)}$$

$$\equiv f_n(t, u_1, \ldots, u_n)$$

(Quasi) Monte Carlo *Explicit sum*

Profumo, Messio, OP, Waintal
PRB 91, 245154 (2015)

- f_n is centered around t. Massive cancellations in the sum.

Interaction expansion of the Green function

Times *Vertices*

$$G_{\uparrow}^{\alpha,\alpha'}(t,t') = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int du_1 du_2 \ldots du_n \left(\prod_{i=1}^{n} U(u_i) \right) \times$$

Keldysh indices

$$\sum_{\alpha_i=\pm 1} \prod_{i=1}^{n} \alpha_i \det M_{\uparrow}(\{u_i\},\{\alpha_i\}) \det P_{\downarrow}(\{u_i\},\{\alpha_i\})$$

$$M_{\sigma}(\{u_k\},\{\alpha_k\}) = \left[\begin{array}{cccc|c} g_{\sigma}^{<}(u_1,u_1) & g_{\sigma}^{\alpha_1\alpha_2}(u_1,u_2) & \ldots & g_{\sigma}^{\alpha_1\alpha_n}(u_1,u_n) & g_{\sigma}^{\alpha_1\alpha'}(u_1,t') \\ \vdots & & & \vdots & \vdots \\ g_{\sigma}^{\alpha_n\alpha_1}(u_n,u_1) & g_{\sigma}^{\alpha_n\alpha_2}(u_n,u_2) & \ldots & g_{\sigma}^{<}(u_1,u_n) & g_{\sigma}^{\alpha_n\alpha'}(u_n,t') \\ \hline g_{\sigma}^{\alpha\alpha_1}(t,u_1) & g_{\sigma}^{\alpha\alpha_2}(t,u_2) & \ldots & g_{\sigma}^{<}(t,u_n) & g_{\sigma}^{\alpha\alpha'}(t,t') \end{array} \right]$$

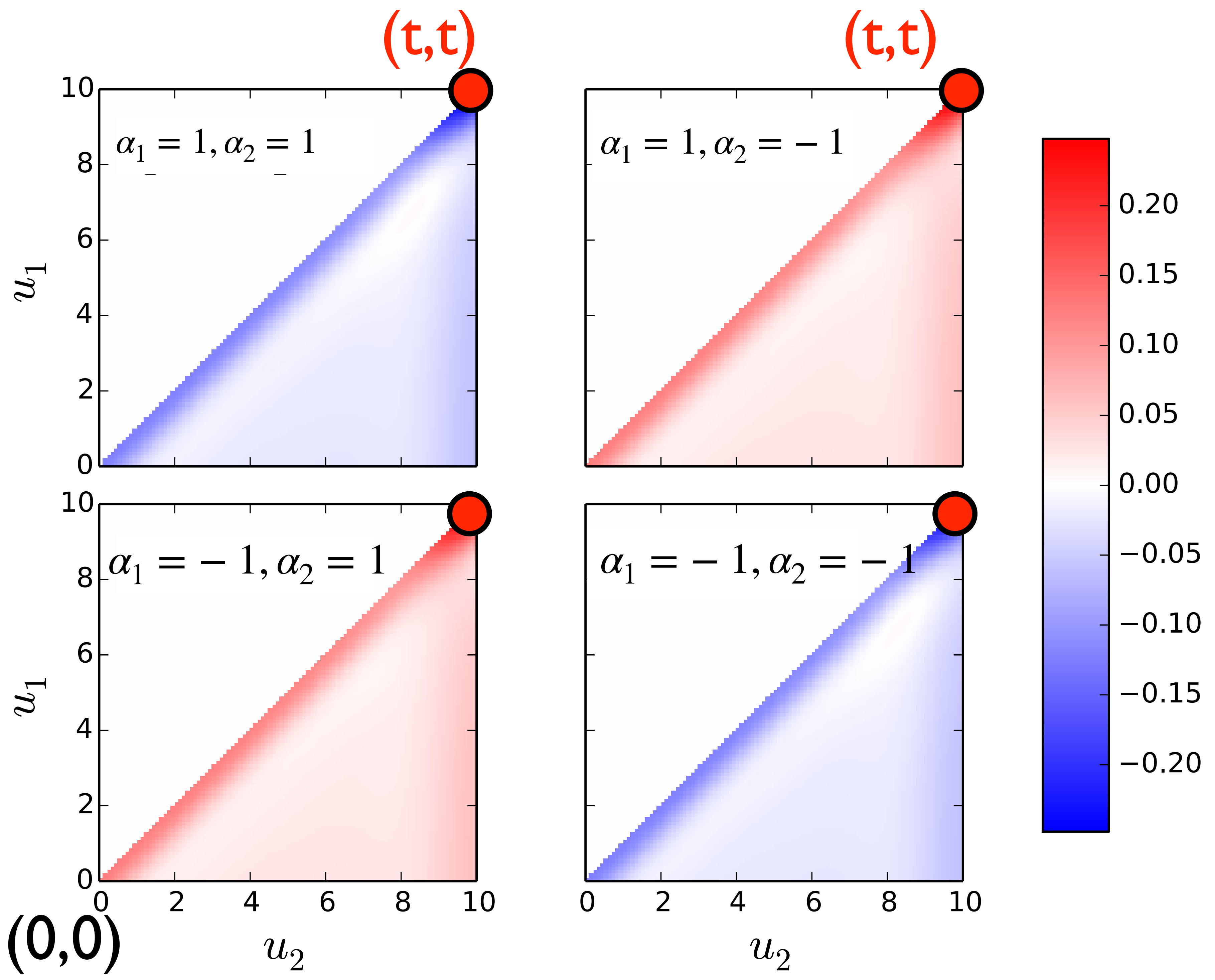
g : U=0
Green function.

$$P_{\sigma}(\{u_k\},\{\alpha_k\}) = \begin{bmatrix} g_{\sigma}^{<}(u_1,u_1) & g_{\sigma}^{\alpha_1\alpha_2}(u_1,u_2) & \ldots & g_{\sigma}^{\alpha_1\alpha_n}(u_1,u_n) \\ \vdots & & & \vdots \\ g_{\sigma}^{\alpha_n\alpha_1}(u_n,u_1) & g_{\sigma}^{\alpha_n\alpha_2}(u_n,u_2) & \ldots & g_{\sigma}^{<}(u_1,u_n) \end{bmatrix}$$

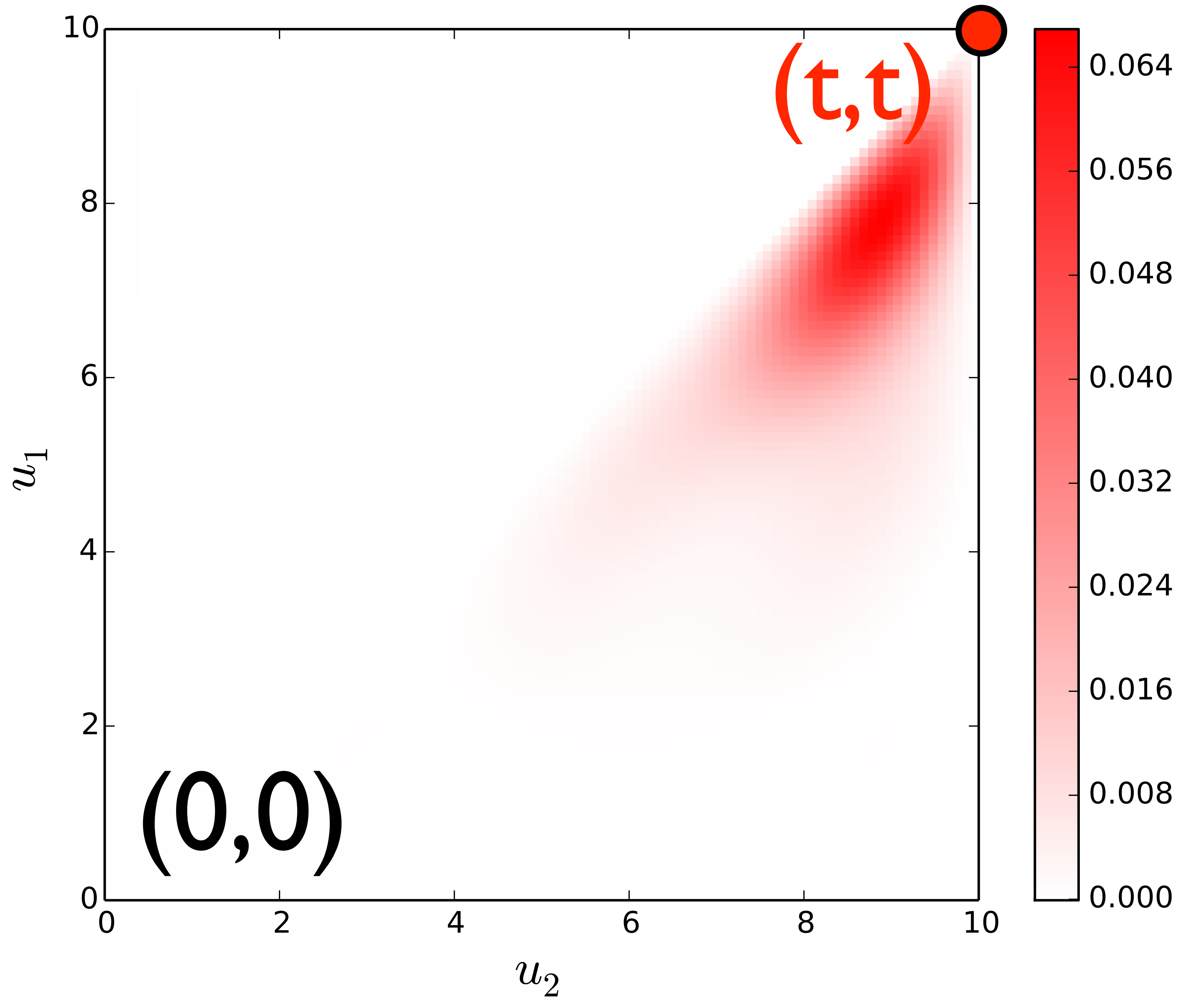
- Integrand cancels except if u_i are close to t = t’

Clusterization around time t=t'. Cancellations. Illustration at n = 2

$$\det M(\alpha_1, \alpha_2, u_1, u_2)$$



$$\sum_{\substack{\alpha_1 = \pm 1 \\ \alpha_2 = \pm 1}} \alpha_1 \alpha_2 \det M(\alpha_1, \alpha_2, u_1, u_2)$$



Profumo, Messio, OP, Waintal
PRB 91, 245154 (2015)

Z=1 Revisited

- Expand Z

$$1 = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int du_1 du_2 \ldots du_n \left(\prod_{i=1}^{n} U(u_i) \right) \times$$

$$\underbrace{\sum_{\alpha_i = \pm 1} \prod_{i=1}^{n} \alpha_i \det P_{\uparrow}(\{u_i\}, \{\alpha_i\}) \det P_{\downarrow}(\{u_i\}, \{\alpha_i\})}_{=0}$$

$$P_\sigma(\{u_k\}, \{\alpha_k\}) = \begin{bmatrix} g_\sigma^<(u_1, u_1) & g_\sigma^{\alpha_1\alpha_2}(u_1, u_2) & \ldots & g_\sigma^{\alpha_1\alpha_n}(u_1, u_n) \\ \vdots & & & \vdots \\ g_\sigma^{\alpha_n\alpha_1}(u_n, u_1) & g_\sigma^{\alpha_n\alpha_2}(u_n, u_2) & \ldots & g_\sigma^<(u_1, u_n) \end{bmatrix}$$

Z=1 Revisited

- Expand Z

$$1 = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int du_1 du_2 \ldots du_n \left(\prod_{i=1}^{n} U(u_i) \right) \times$$

$$\underbrace{\sum_{\alpha_i = \pm 1} \prod_{i=1}^{n} \alpha_i \det P_\uparrow(\{u_i\}, \{\alpha_i\}) \det P_\downarrow(\{u_i\}, \{\alpha_i\})}_{=0}$$

- Proof : For fixed u_i, cancellation. Take u_{max} the largest u_i.

$$g^{\alpha_i +}(u_i, u_{\max}) = g^{\alpha_i -}(u_i, u_{\max}) \qquad \forall i$$

$$g^{+\alpha_i}(u_{\max}, u_i) = g^{-\alpha_i}(u_{\max}, u_i) \qquad \forall i$$

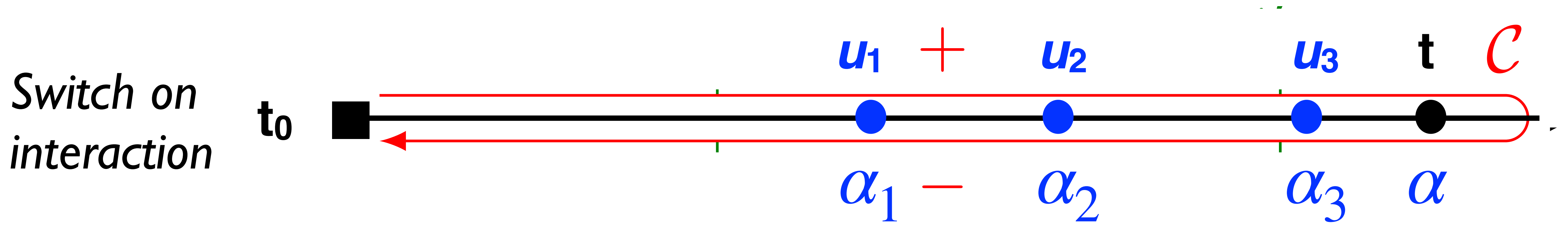
$$G_{ab}^{++}(t,t') = \theta(t-t')G_{ab}^{-+}(t,t') + \boxed{\theta(t'-t)G_{ab}^{+-}(t,t')}$$

$$G_{ab}^{--}(t,t') = \boxed{\theta(t'-t)G_{ab}^{-+}(t,t')} + \theta(t-t')G_{ab}^{+-}(t,t')$$

- The dets do not depend on α_{max}, so it cancels the sum.

How to compute $Q_n(t)$?

- Schwinger-Keldysh formalism
  Q_n is a n-dimensional integral



Vertices. Times u_i.
Keldysh indices $\alpha = -1, 1$

$$Q_n(t) = \frac{1}{n!}\boxed{\int_{t_0}^{\infty} du_1 \ldots du_n}\;\boxed{\left(\sum_{\alpha_i=\pm 1}\prod_i \alpha_i \ \det(\ldots)\right)}$$

$$\equiv f_n(t, u_1, \ldots, u_n)$$

(Quasi) Monte Carlo *Explicit sum*

Profumo, Messio, OP, Waintal
PRB 91, 245154 (2015)

- Long time limit t→∞ is easy. *f_n* is centered around t. Massive cancellations in the sum.
- O(2^n) cost to compute f_n(u). In practice, n = 10-15.

$$Q(\underset{\substack{\uparrow \\ \text{Time}}}{t}, U) = \sum_{n=0}^{K} Q_n(t)\,U^{n}\underset{\nwarrow\ \text{Interaction}}{}$$

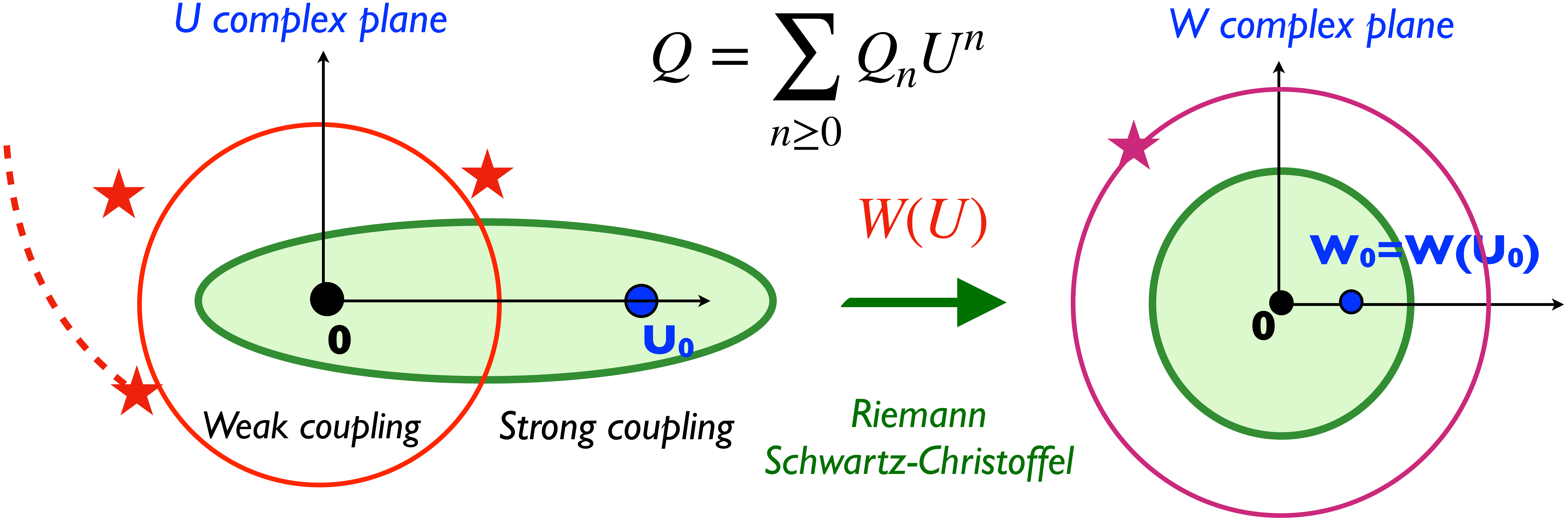
How to sum the series ?

Using the perturbative series: three possibilities

1. At finite time ***t***, the series is convergent *Bertrand et al. Phys. Rev. X 9, 041008 (2019)*
2. A infinite ***t*** (steady state), the series has a finite radius of convergence (for impurity, lattice models). Need re-summation technique
3. Change the starting point, cf M. Ferrero's talk, see also *Profumo et al. PRB 91, 245154 (2015)*

Resum with conformal maps

Profumo et al. PRB 91, 245154 (2015)
Bertrand et al. Phys. Rev. X 9, 041008 (2019)



U complex plane

$$Q = \sum_{n \geq 0} Q_n U^n$$

W complex plane

$W(U)$

0

U₀

W₀=W(U₀)

0

Weak coupling

Strong coupling

Riemann
Schwartz-Christoffel

A finite radius of convergence !
Singularities poles, branch cuts

- Change of variable W(U), with W(0) = 0

$$Q = \sum_{n \geq 0} Q_n U^n = \sum_{p \geq 0} \bar{Q}_p W^p$$

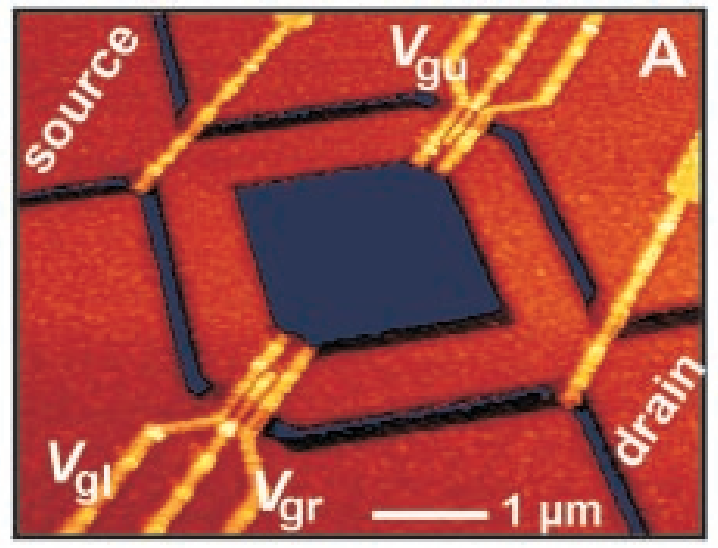
Converges at W₀

Let us end with some results
(quantum dot)

1- Equilibrium. Benchmarks.

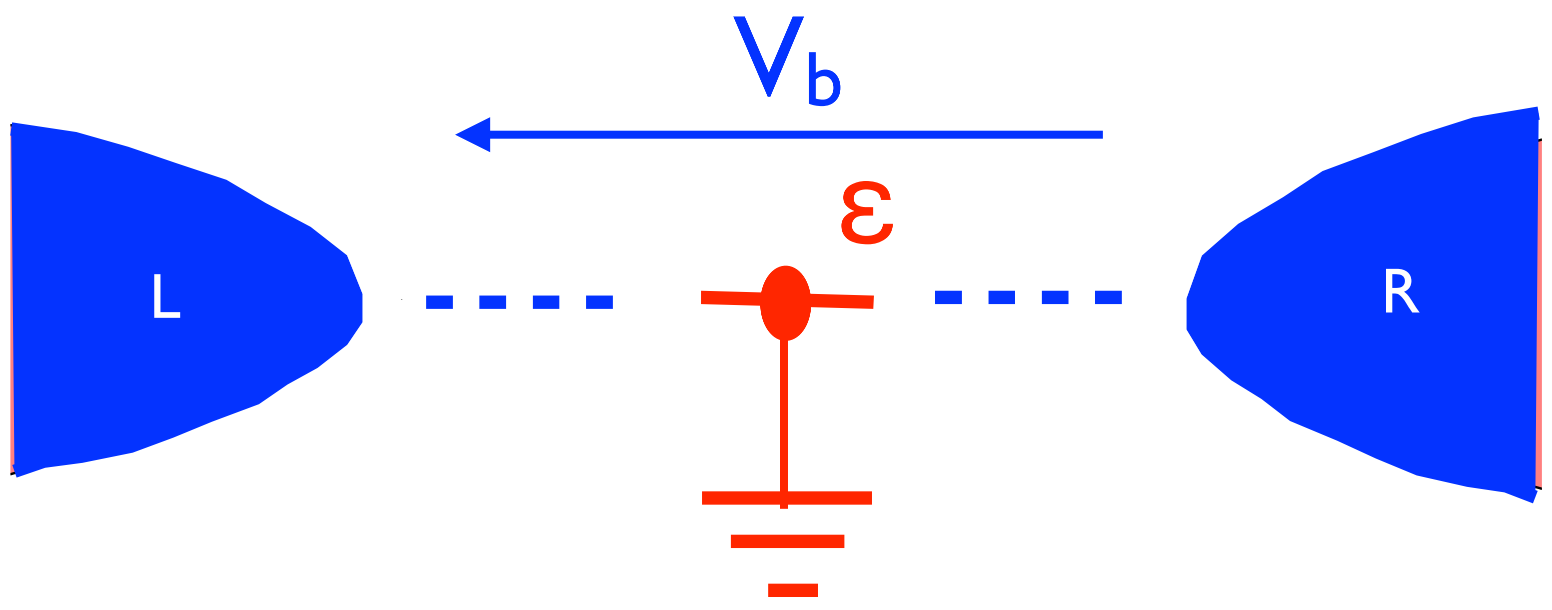
Reminder : model for the quantum dot

- Anderson model with two leads (L, R).



Bath *Local orbital* *Hybridization*

$$H = \sum_{\substack{k\sigma \\ \alpha=L,R}} \varepsilon_{k\alpha} c^\dagger_{k\sigma\alpha} c_{k\sigma\alpha} + \sum_{\sigma} \varepsilon_d d^\dagger_\sigma d_\sigma + U n_{d\uparrow} n_{d\downarrow} + \sum_{\substack{k\sigma \\ \alpha=L,R}} g_{k\sigma\alpha} (c^\dagger_{k\sigma\alpha} d_\sigma + h.c.)$$



Level width at U=0

$$\Gamma = \pi \rho_{E_F} g^2$$

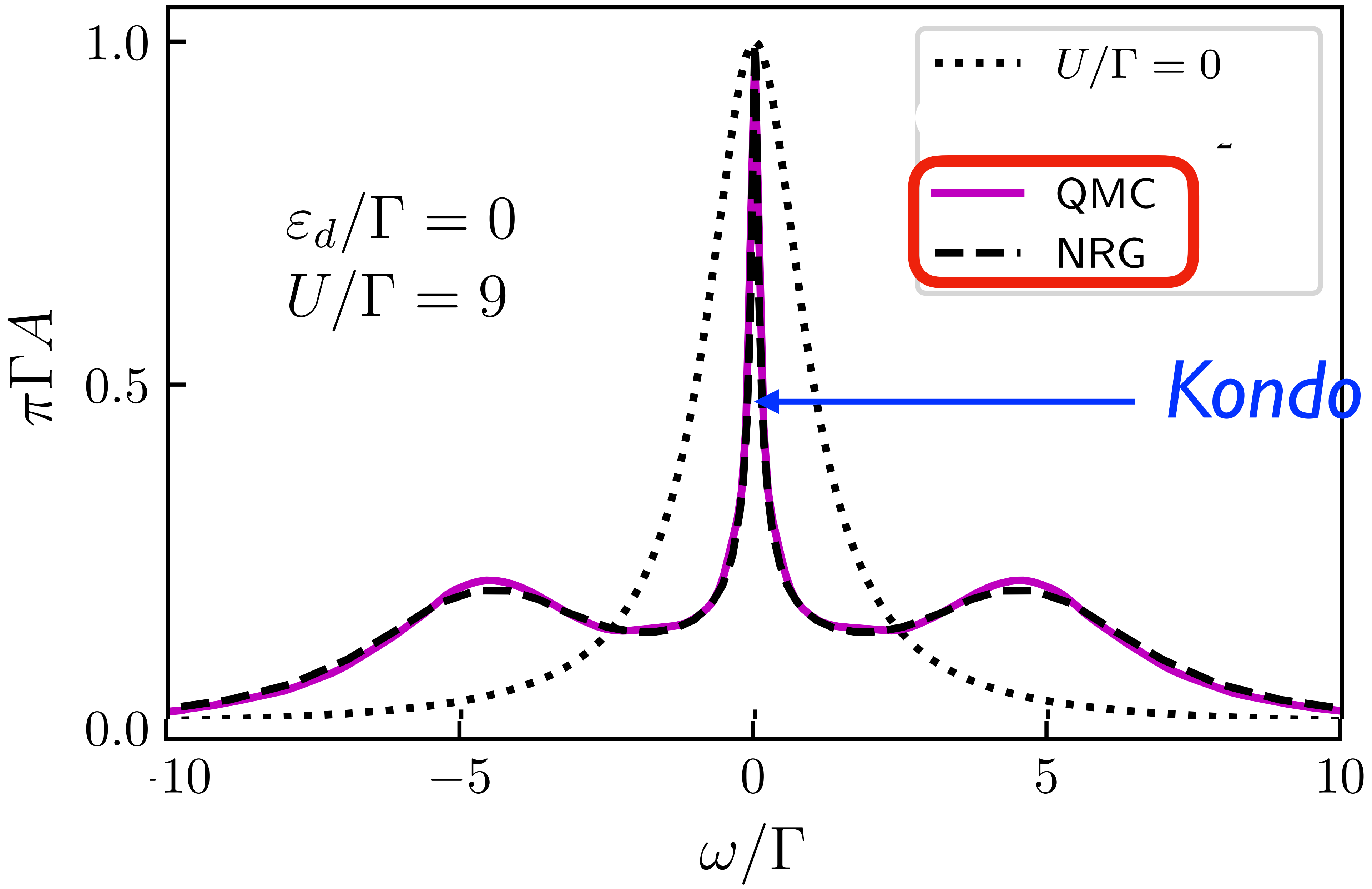
- Questions : Spectral function ? Kondo temperature ? Current ?

We want a precise solution, at low temperature, any V_b, <u>in steady state</u>

Kondo effect in equilibrium

$$A(\omega) = -\frac{1}{\pi} Im G^R(\omega)$$

Spectral function on the dot



Kondo resonance

- Sum the series for each frequency independently
- Resumption of the series using conformal maps
- Benchmark with NRG (numerical renormalisation group)

$T = 10^{-4}\Gamma$

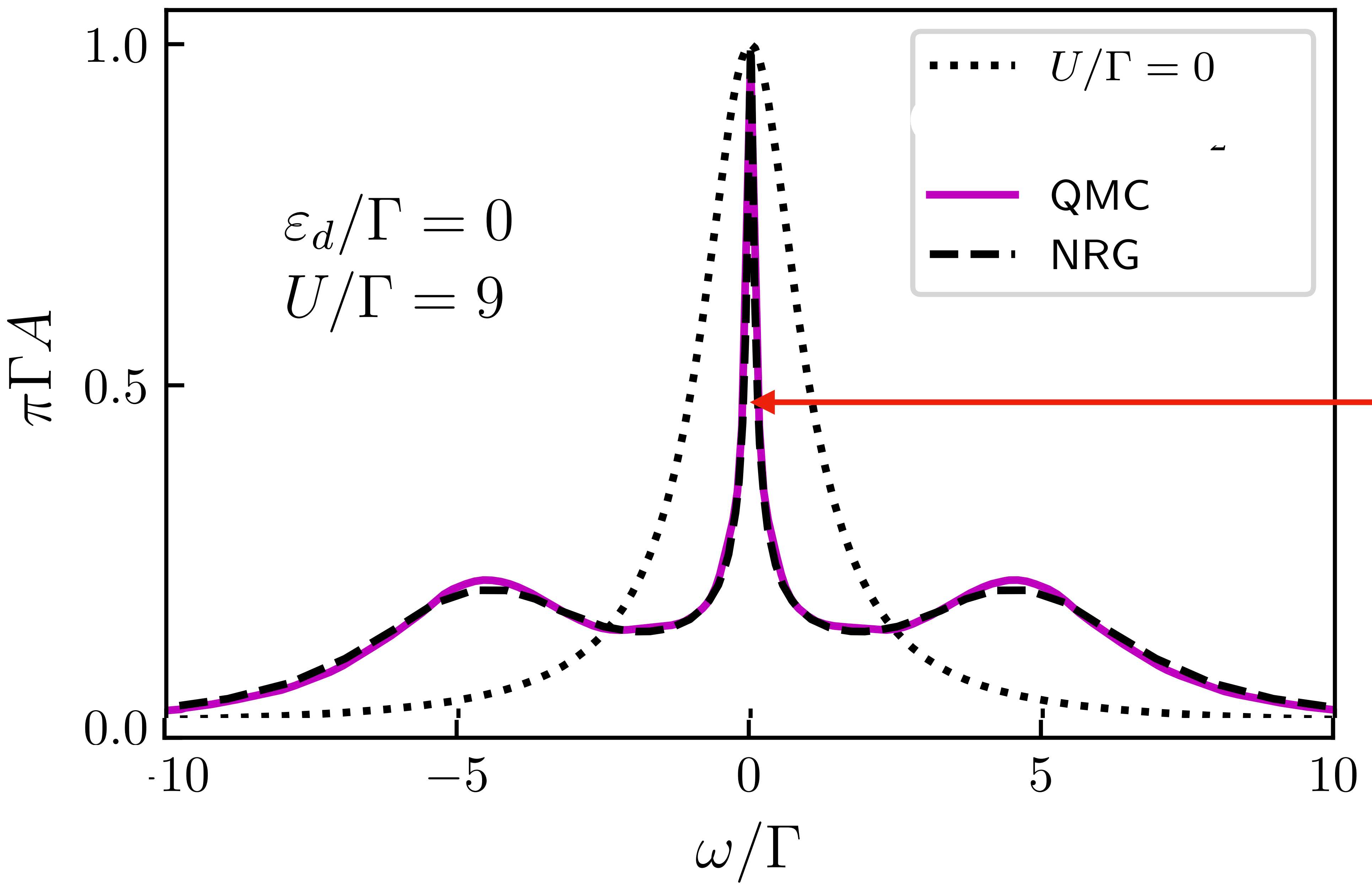
C. Bertrand et al.
Phys. Rev. X 9, 041008 (2019)

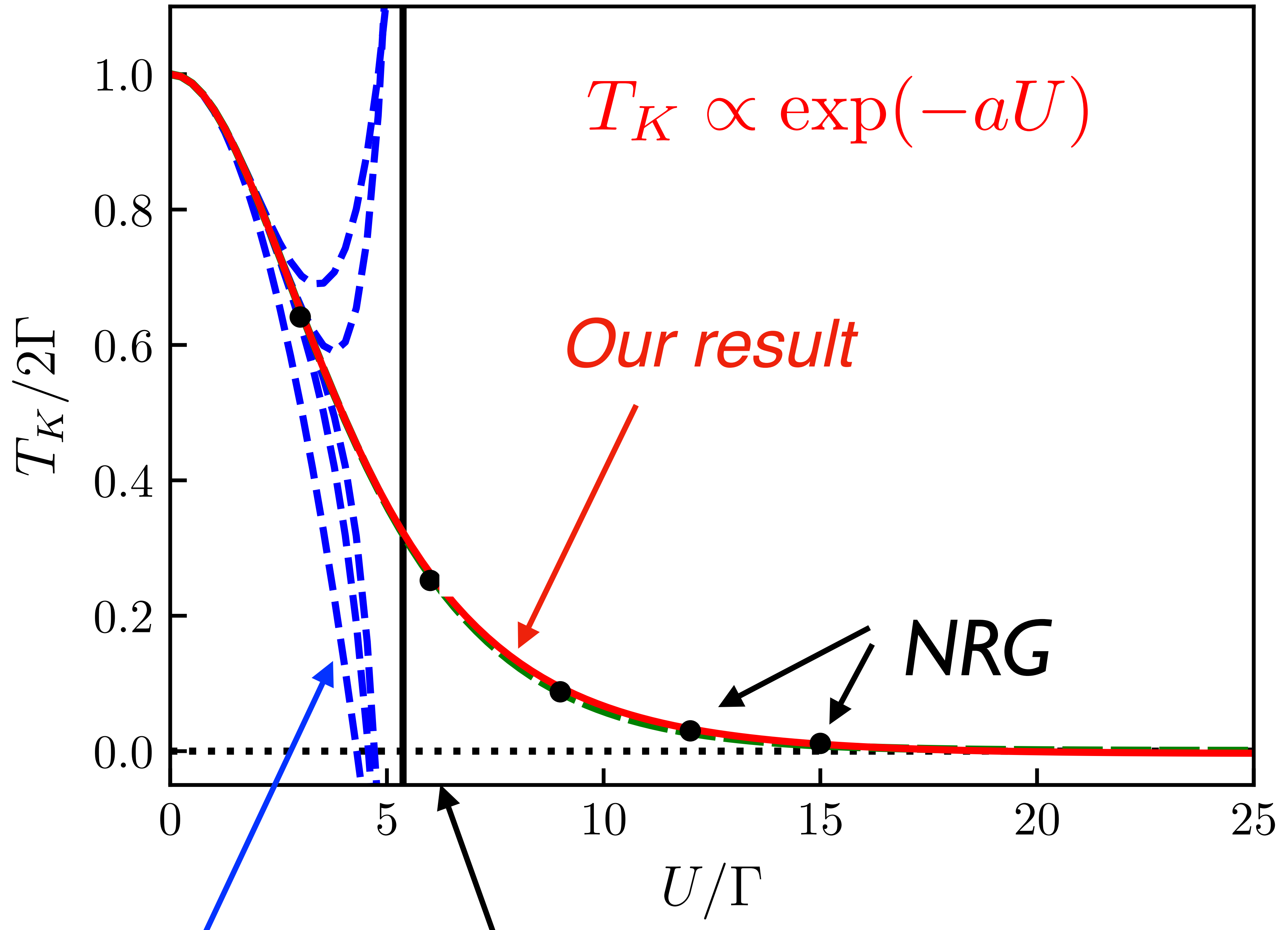
Kondo Temperature

$$T_K(U) \equiv \frac{2\Gamma}{1 - \partial_\omega \mathrm{Re}\Sigma^R(U,\omega)\Big|_{\omega=0}}$$

C. Bertrand et al.
Phys. Rev. X 9, 041008 (2019)

Kondo temperature





Our result

NRG

Naive sum of the series

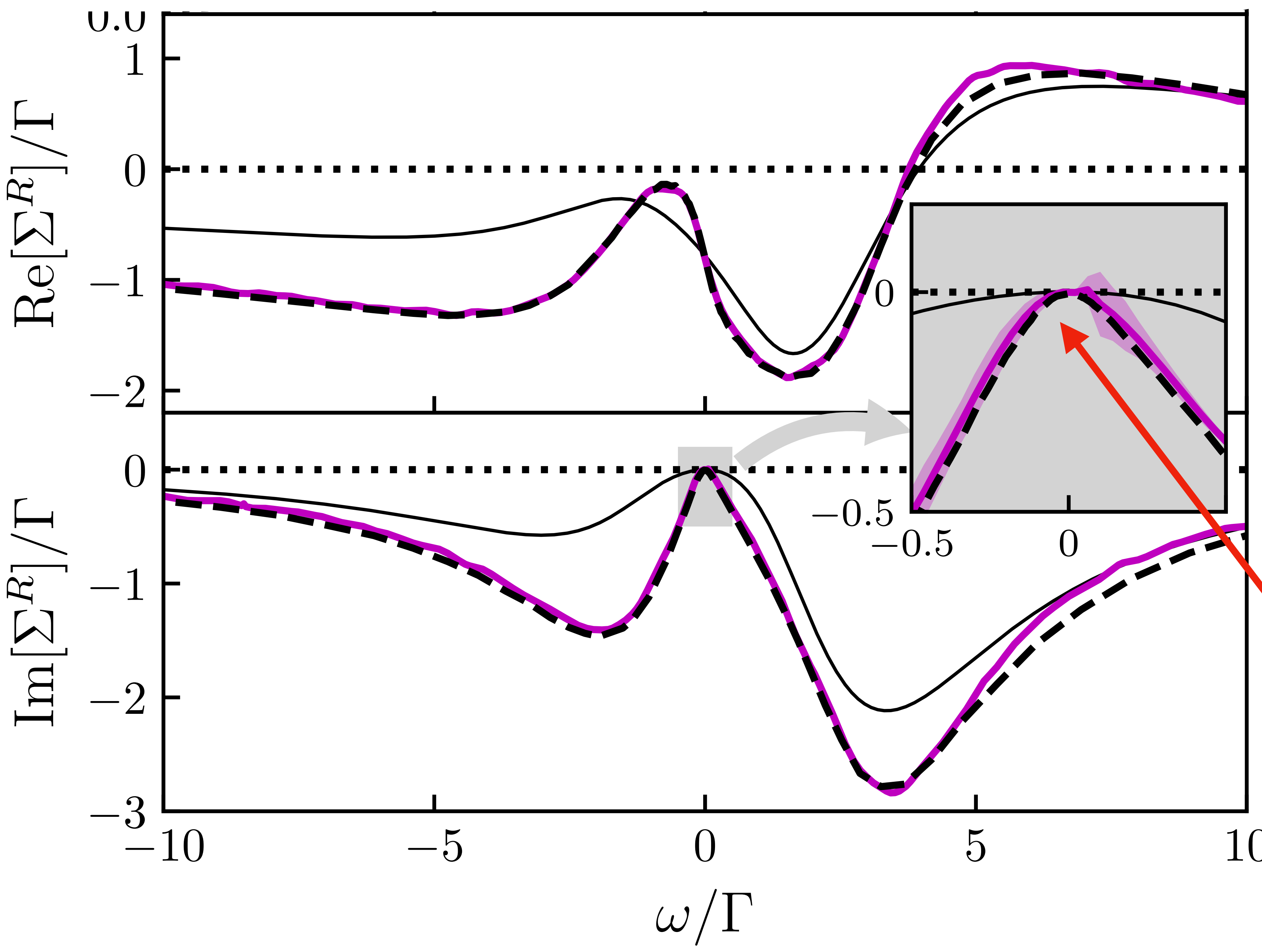
Convergence radius

Fermi liquid at low energy

- Equilibrium. Self-energy, away from particle-hole symmetry

Self energy (Re)

Self energy (Im)



Our result

NRG

$\epsilon_d/\Gamma = 1$

$U/\Gamma = 6$

$T = 0$

$Im\Sigma(\omega) \sim \omega^2$

C. Bertrand et al.
Phys. Rev. X 9, 041008 (2019)

Benchmarks

- Steady state inchworm
  by A. Erpenbeck et al.

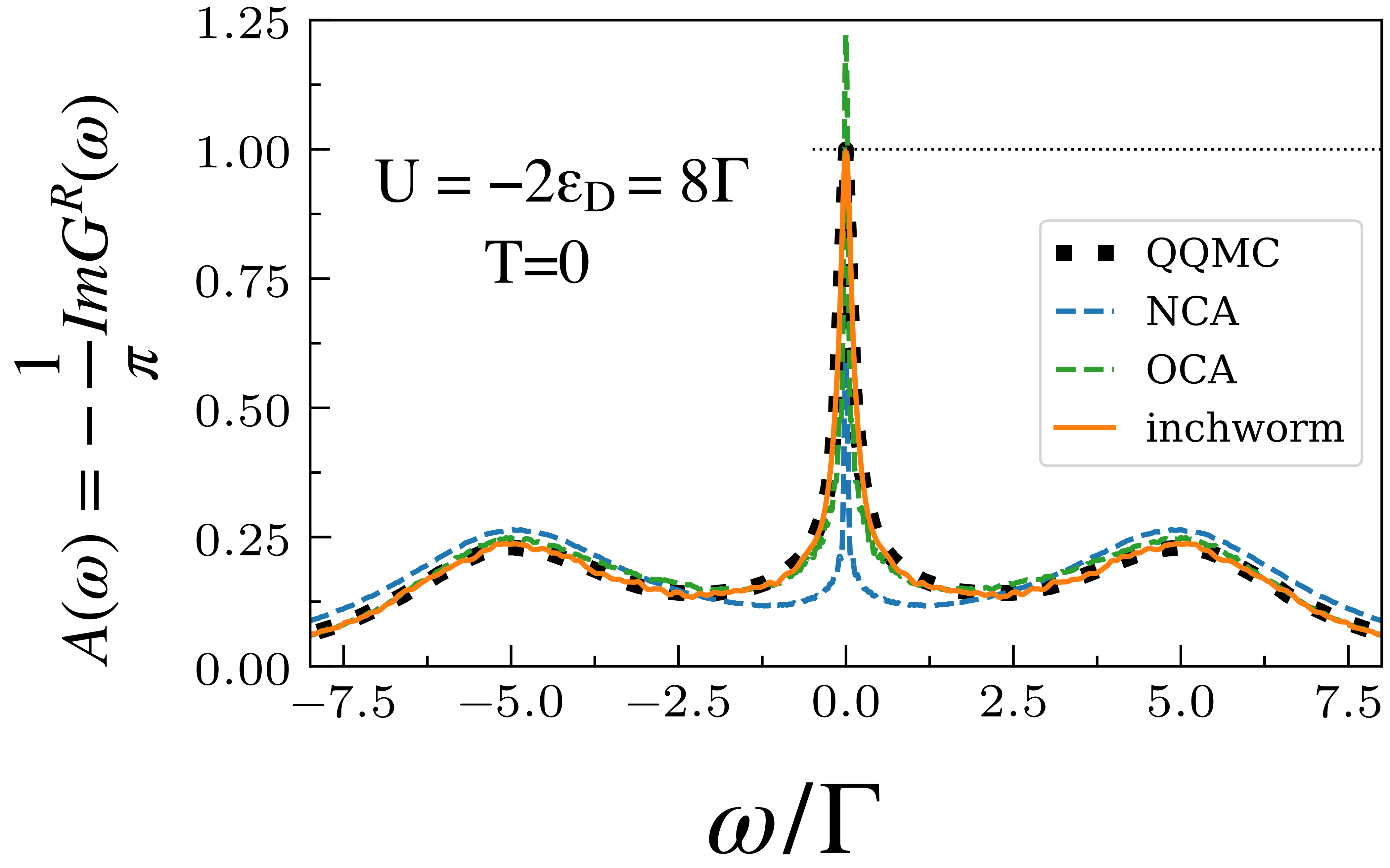
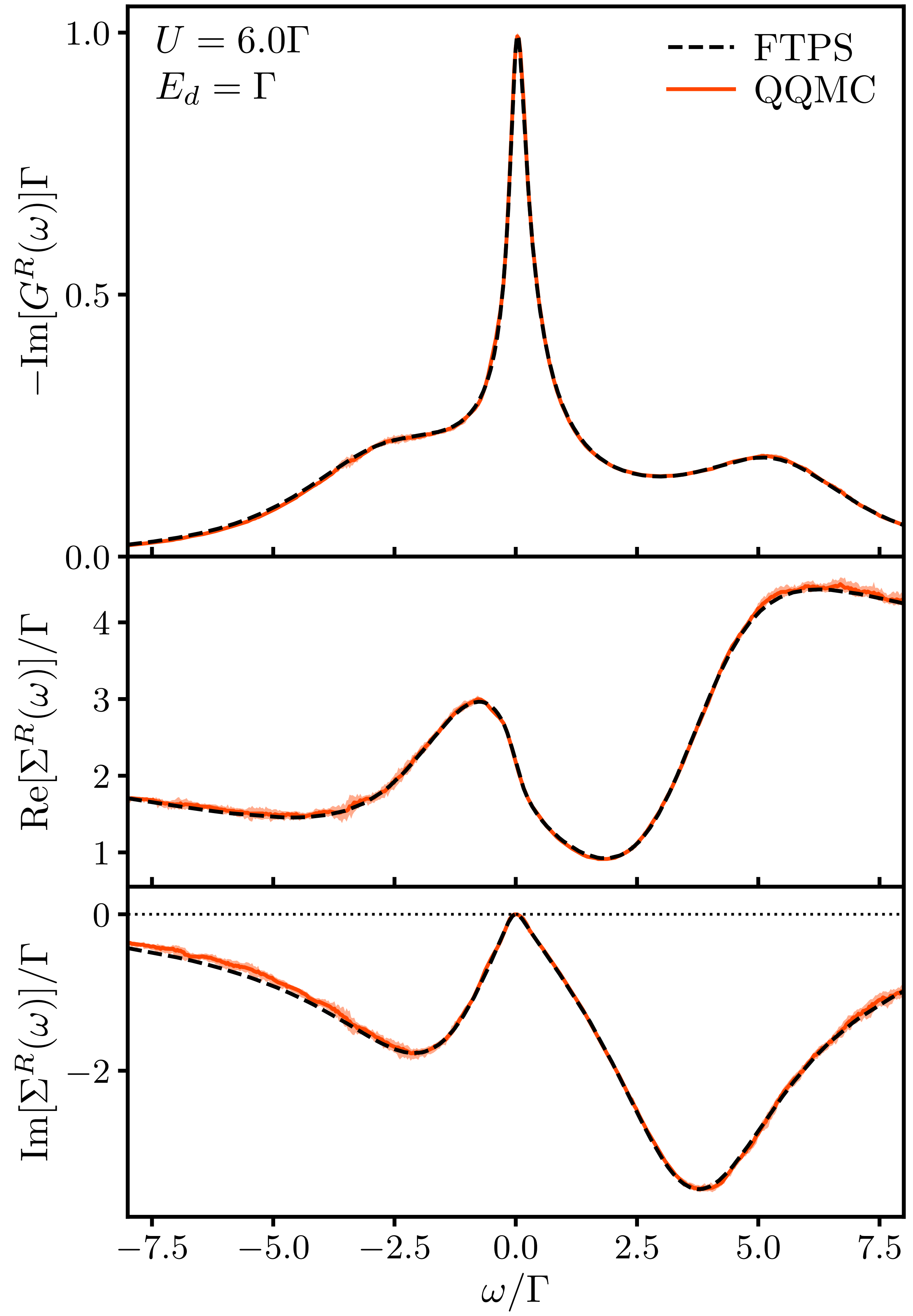


Figure from A. Erpenbeck

- Tensor network (MPS) + time evolution



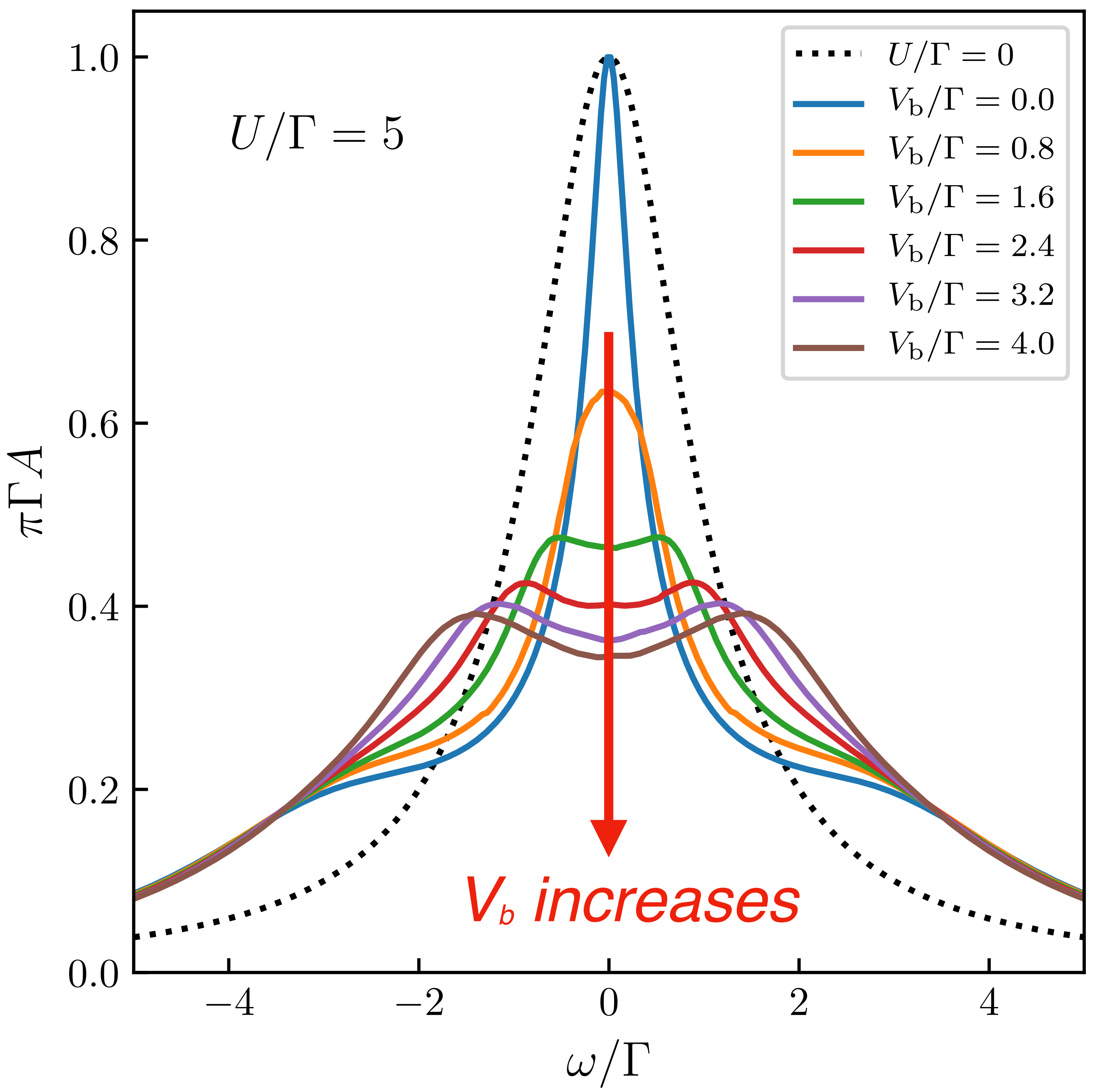
C. Bertrand, D. Bauernfeind, P. Dumitrescu, M. Maček,
*X. Waintal, **O.P.***
Phys. Rev. B 103, 155104 (2021)

2-Non equilibrium

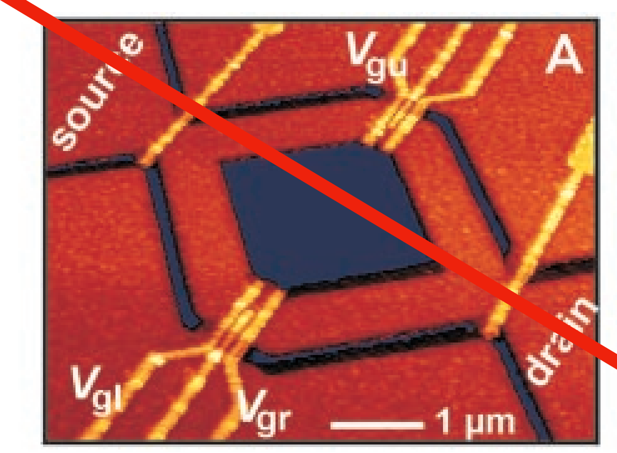
Out of equilibrium
Spectral function

Bertrand et al. 2019
Phys. Rev. X *9, 041008 (2019)*

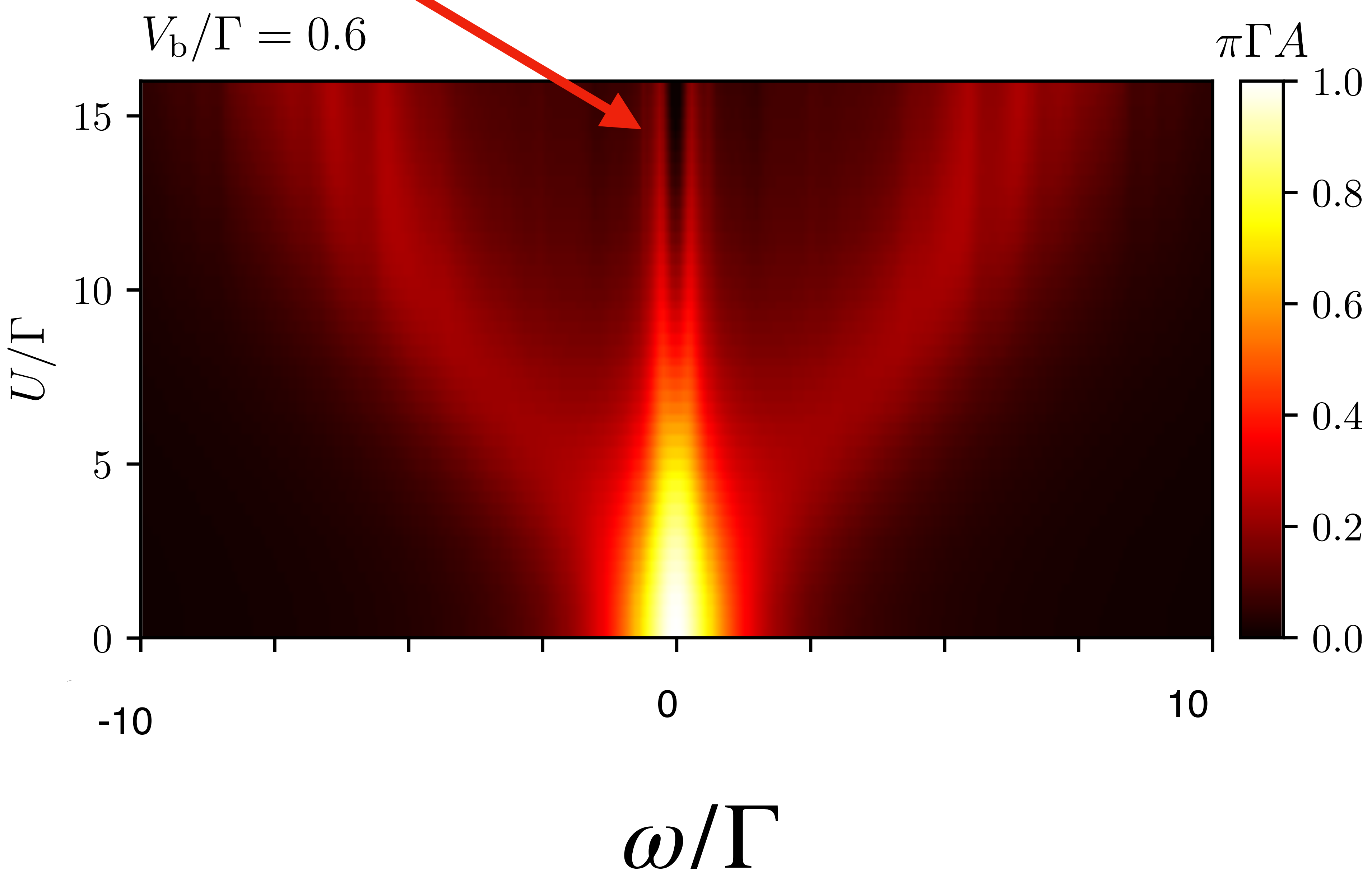
- Destruction of the Kondo resonance by voltage bias



V_b *increases*



V_b



T = 0

T = Γ/50

I-V_b Characteristics

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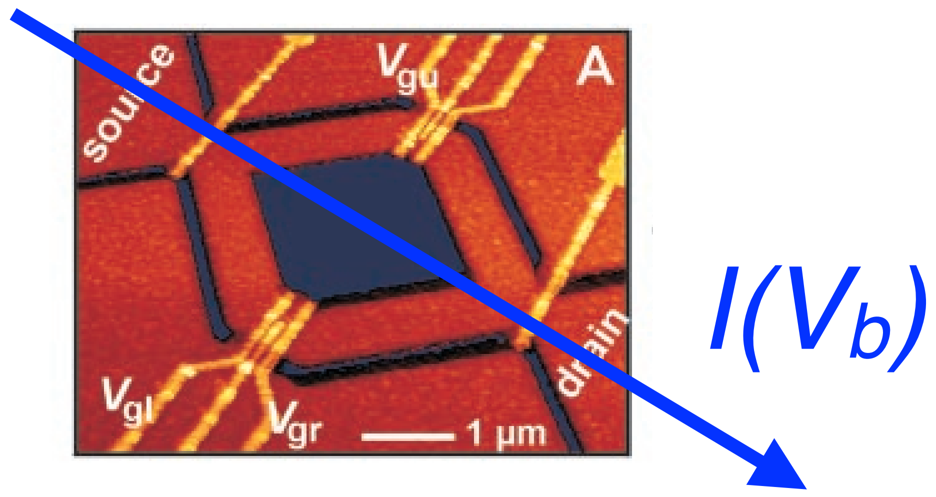
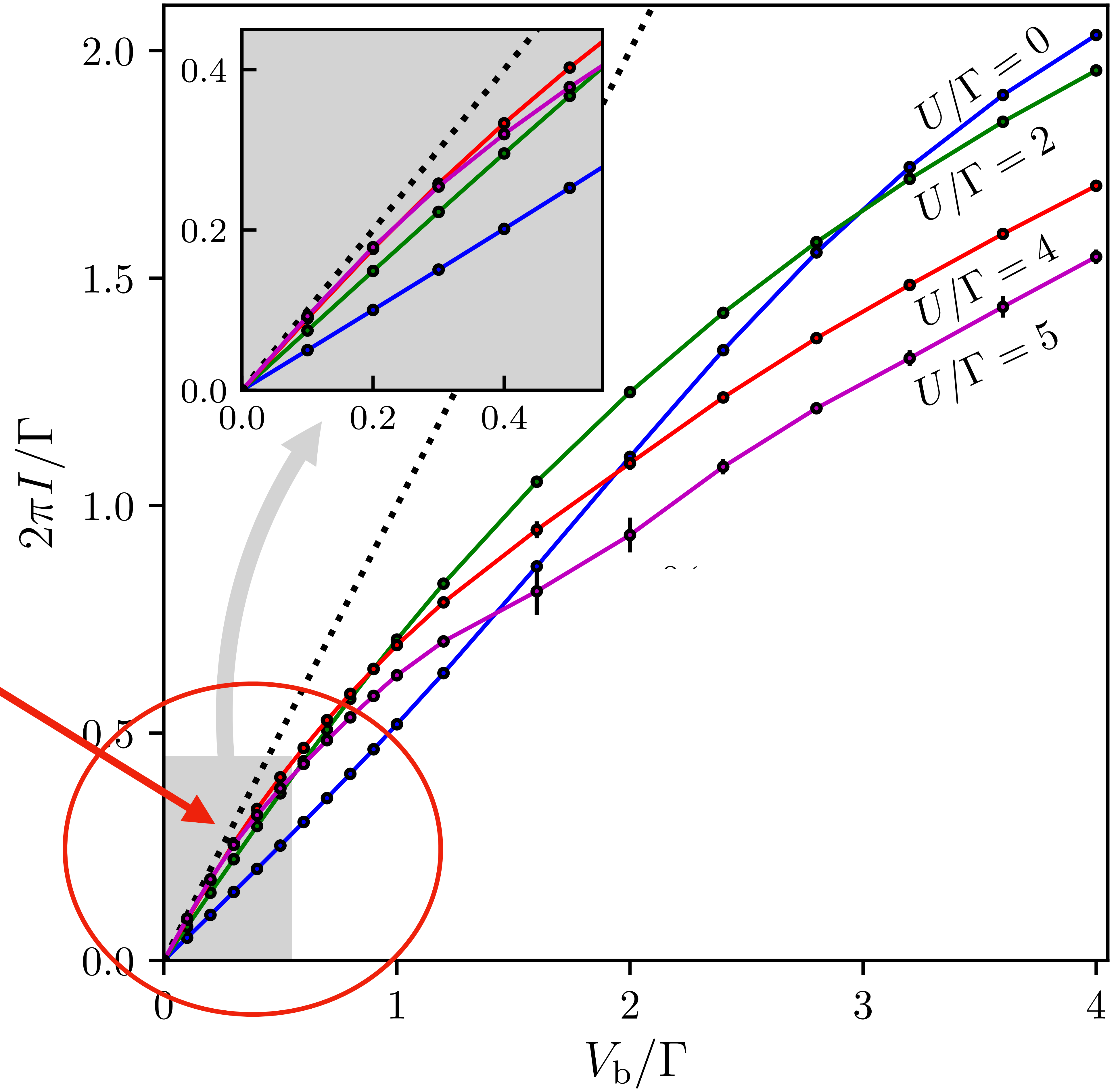
***Phys. Rev. X** 9, 041008 (2019)*

- Particle hole ***asymmetric*** case



$$G \sim \frac{e^2}{h}$$

Kondo effect
Unitary limit



Distribution function on the dot

- Equilibrium, for all U. Fermi function.

$$f(\omega) = n_F(\omega - \mu_R) = \frac{1}{1 + e^{\beta(\omega - \mu_R)}}$$

- Out of equilibrium for U = 0 (in the small g limit). Not a Fermi function

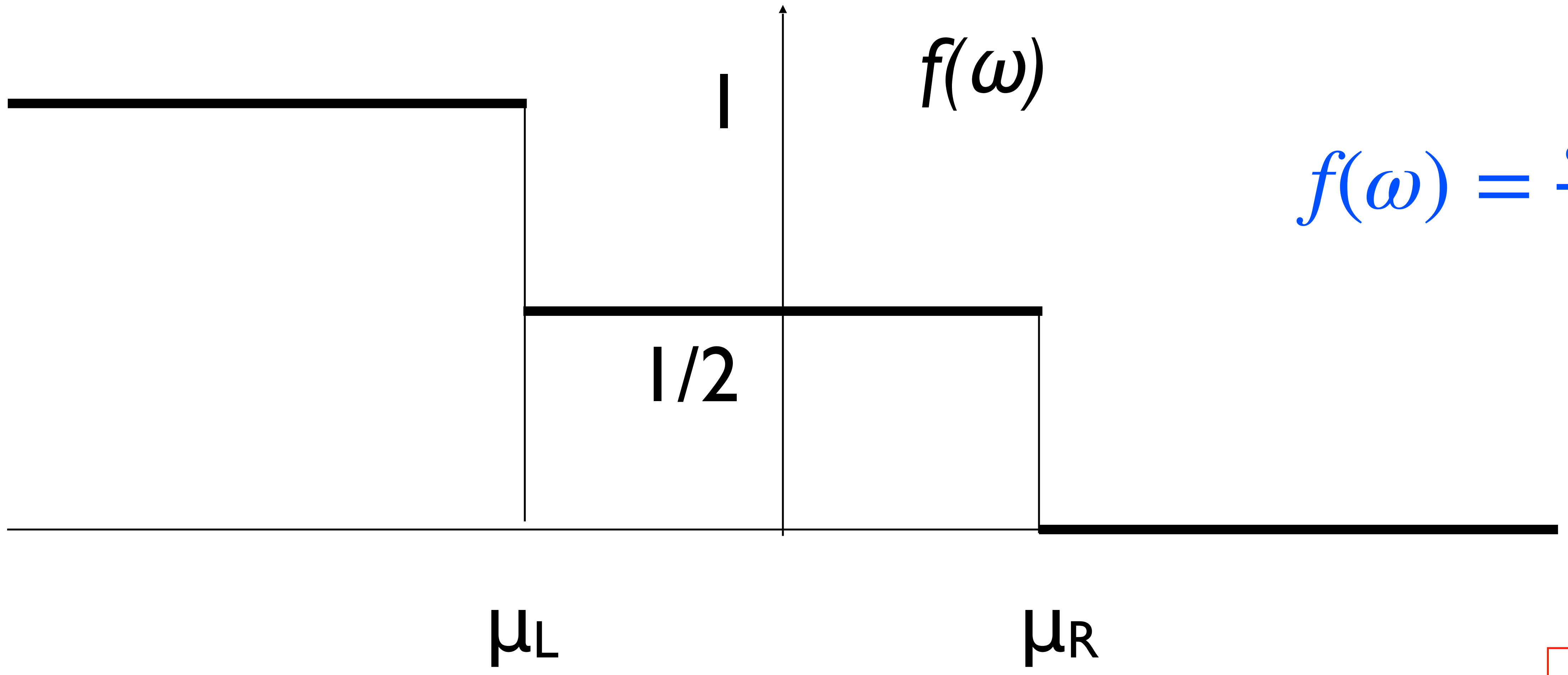


Illustration at T=0, $g_L = g_R$

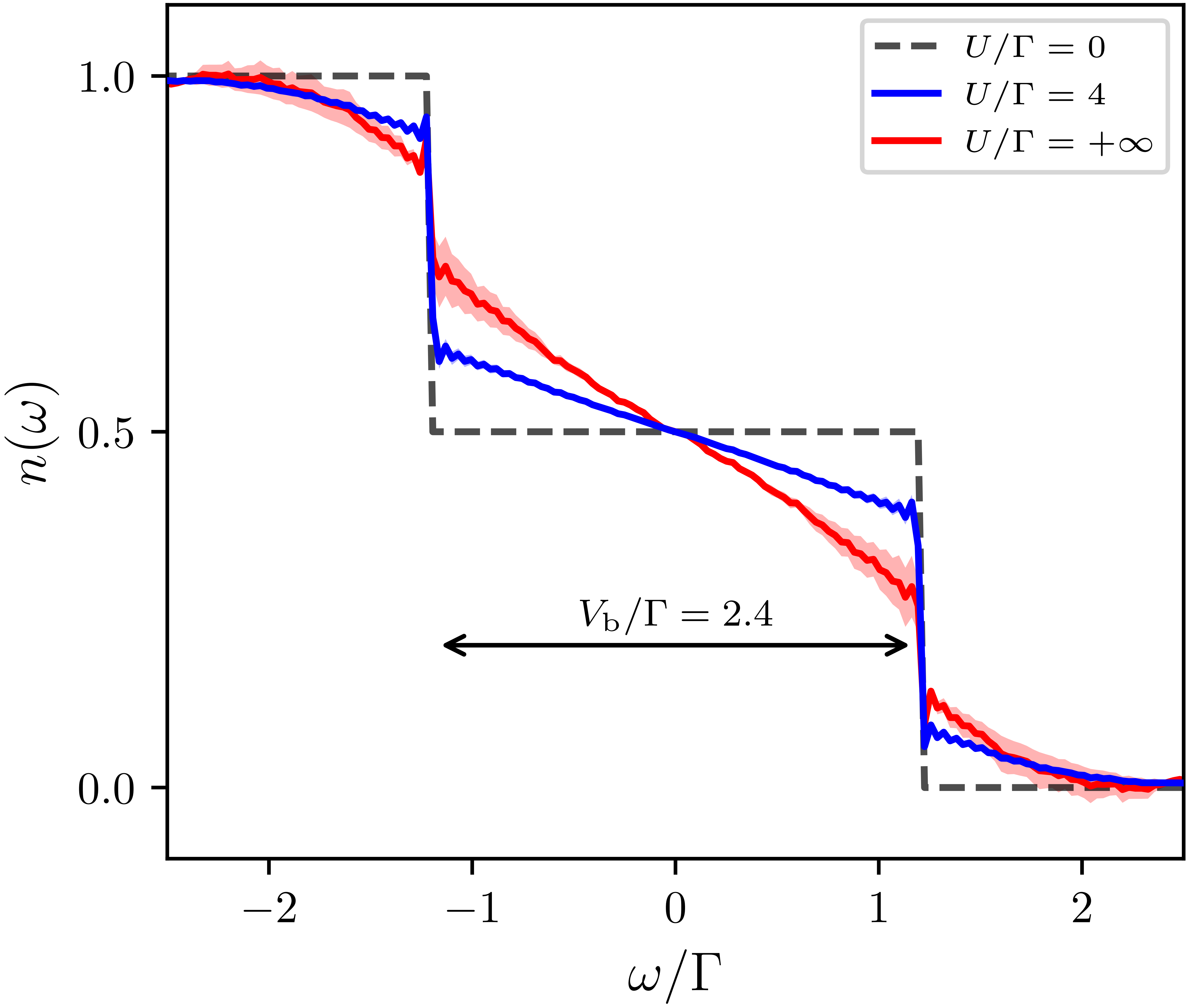
$$f(\omega) = \frac{g_L^2 n_F(\omega - \mu_L) + g_R^2 n_F(\omega - \mu_R)}{g_L^2 + g_R^2}$$

Effect of interaction ?

Out of equilibrium distribution function of the dot

- Finite U, T=0

$$n(\omega) \equiv \frac{G^{<}(\omega)}{2\pi i A(\omega)}$$



Bertrand et al. 2019
***Phys. Rev. X** 9, 041008 (2019)*

Conclusion

- Perturbation theory for real time/out of equilibrium systems.
- Success in quantum dots/nano-electronic systems.
- Beyond Monte-Carlo …
- Next steps : lattice, DMFT solver out of equilibrium.

References

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Thank you for your attention!