Topological Entanglement & Quantum Computation

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BIRS Spin, Charge, and Topology in low dimensions workshop

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Outline:

Motivation of Topological QC

Two models of Topological QC

An entanglement theoretic study of Topological QC

Some related subjects

a major problem in QC: quantum information is more error-prone than classical information

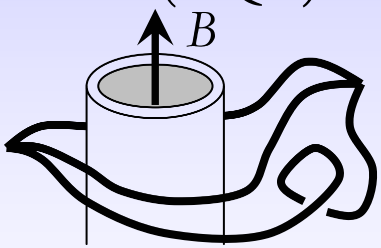
controlling errors ⟶ **decoherence**

Remedy: error-correction methods & other fault-tolerant schemes

Fault-tolerant QC: encoding quantum information in degrees of freedom that are error-resilient

⟶ **Topological Quantum Computation** (TQC)

Aharonov-Bohm effect

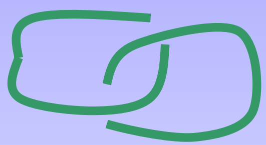


Abelian & non-Abelian **anyons** (FQHE)

another motivation: new powerful quantum algorithms topological entanglement $\longleftrightarrow$ quantum entanglement

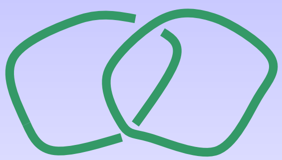
Aravind's hypothesis:

linked knots $\overset{?}{\longleftrightarrow}$ entangled states

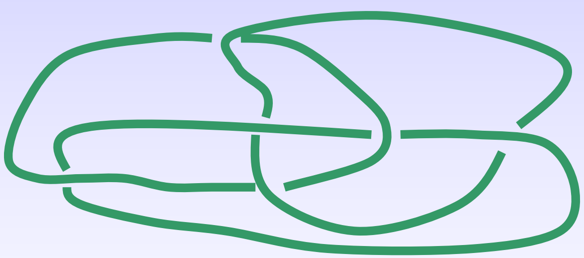


(Hopf link)

$$\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$



$$|\Phi\rangle \otimes |\Psi\rangle$$



(Borromean link)

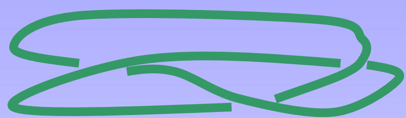
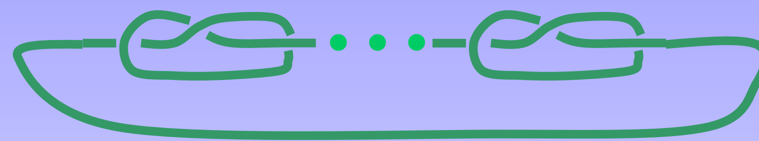
$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

$$|\Lambda\rangle = \frac{1}{2}\left(|000\rangle + |110\rangle + |101\rangle + |011\rangle\right)$$

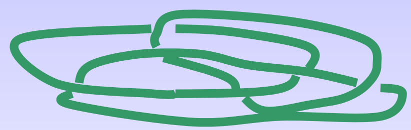
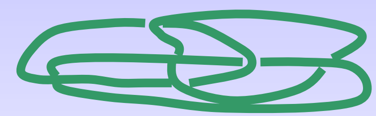
<u>Issues</u>: Inconsistencies

❶ particle $\overset{?}{\longleftrightarrow}$ link component

topological self-entanglement


(trefoil),


❷ local invariance of entanglement

$$|\Lambda\rangle = H \otimes H \otimes H |GHZ\rangle$$

$\not\equiv$


❸ measurement $\overset{?}{\longleftrightarrow}$ cutting a link

post-measurement state depends on <u>measurement basis & result</u>

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \xrightarrow{\{|0\rangle,|1\rangle\}} \begin{array}{ll} |0\rangle: & \text{entangled} \\ |1\rangle: & \text{separable} \end{array}$$

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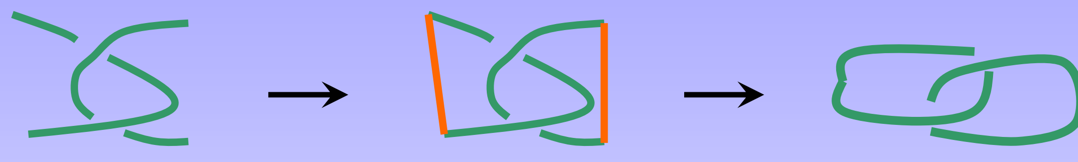
$\Longrightarrow$ knots $\longleftrightarrow$ states **Inconsistent!**

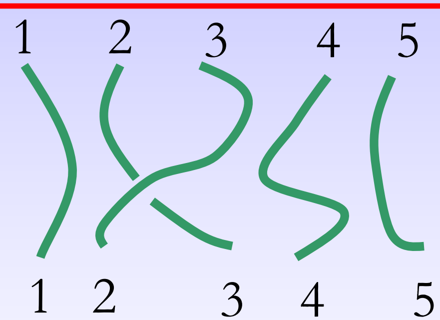
<u>second approach</u>: motivated by **Alexander's Theorem**

Theorem: Any knot can be obtained from closure of some braid



braids $\overset{?}{\longleftrightarrow}$ entangling operators

Braid generators:



$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |i-j|>1$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad i=1,\ldots,n-2$$

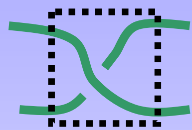
<u>Yang-Baxter equation</u>: $R: \quad V \otimes V \rightarrow V \otimes V$

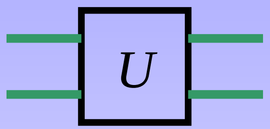
$$\tilde{R}_{12}\tilde{R}_{13}\tilde{R}_{23} = \tilde{R}_{23}\tilde{R}_{13}\tilde{R}_{12} \qquad (R = S.\tilde{R}) \longrightarrow \text{ solutions of YBE?}$$

$V = \mathbb{C}^2 \longrightarrow$ a full classification of solutions (Hietarinta, Hlavaty, Dye)

Now, braiding operators (solutions of YBE) can be considered as 2-qubit quantum operators.

Question: Can braiding operators be used for QC?

topological 



quantum mechanical

Are braiding operators sufficient for universal QC?

Universality Theorem: 1-qubit unitaries + any entangling 2-qubit unitary

$$U \in \mathrm{SU}(4) \qquad \exists |\alpha\rangle, |\beta\rangle: \quad U(|\alpha\rangle \otimes |\beta\rangle) \neq |\alpha'\rangle \otimes |\beta'\rangle$$

E.g.: CNOT $\quad |x\rangle \otimes |y\rangle \rightarrow |x\rangle \otimes |x \oplus y\rangle$

Local equivalence:

$$U \equiv U' \quad \Leftrightarrow \quad \exists u_1, v_1, u_2, v_2 \in \mathrm{SU}(2); \qquad U' = (u_1 \otimes v_1) U (u_2 \otimes v_2)$$

Theorem: Canonical decomposition $U \in SU(4)$

$$U = (u_1 \otimes v_1) \underbrace{e^{-i(c_1\sigma_x\otimes\sigma_x + c_2\sigma_y\otimes\sigma_y + c_3\sigma_z\otimes\sigma_z)}}_{\text{Heisenberg interaction}} (u_2 \otimes v_2)$$

⟶ classification of local equivalence: $\begin{cases} U \equiv [c_1, c_2, c_3] \\ G_i[U]: \ \text{local invariants} \end{cases}$

- If we can find an entangling gate among YBE solutions, then braiding operators are able to do universal QC.

⟶ selection of **unitary** solutions of YBE: (by a brute-force search)

(up to some symmetries & local equivalence)

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & 1 \\ & 1 & -1 & \\ & 1 & 1 & \\ -1 & & & 1 \end{pmatrix} \equiv \text{CNOT} \qquad R' = \begin{pmatrix} a_1 & & & \\ & & a_2 & \\ & a_3 & & \\ & & & a_4 \end{pmatrix} \quad (|a_i| = 1)$$

$a_i = 1 \Rightarrow \text{SWAP}$

$a_{i\neq 3} = 1, a_3 = -1 \Rightarrow \text{DCNOT}$

⇨ **universality** ✓

Question: How entangling braid operators are?!

quantification of **entangling power**: $e_p(U) := \overline{E(U\,|\psi\rangle \otimes |\varphi\rangle)}^{|\psi\rangle,|\varphi\rangle}$

- $\overline{}^{|\psi\rangle,|\varphi\rangle}$: average on the manifold of separable states with the probability p
- E : an entanglement measure for states, e.g.: linear entropy $E(|\Psi\rangle_{AB}) := 1 - \mathrm{tr}(\rho_A{}^2)$

- Entangler: $\exists |\alpha\rangle, |\beta\rangle: \quad U(|\alpha\rangle \otimes |\beta\rangle) \neq |\alpha'\rangle \otimes |\beta'\rangle$ ⟶ entangled

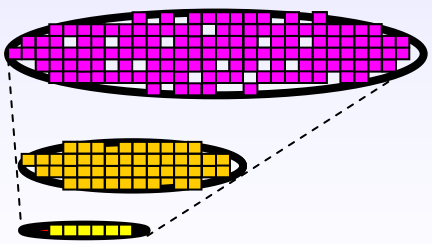
  entangling some separable state

- Perfect entangler: $\exists |\alpha\rangle, |\beta\rangle: \quad U(|\alpha\rangle \otimes |\beta\rangle) \neq |\Psi_{\alpha\beta}\rangle$ ⟶ maximally entangled

  E.g.: CNOT, DCNOT, $\sqrt{\text{SAWP}}$

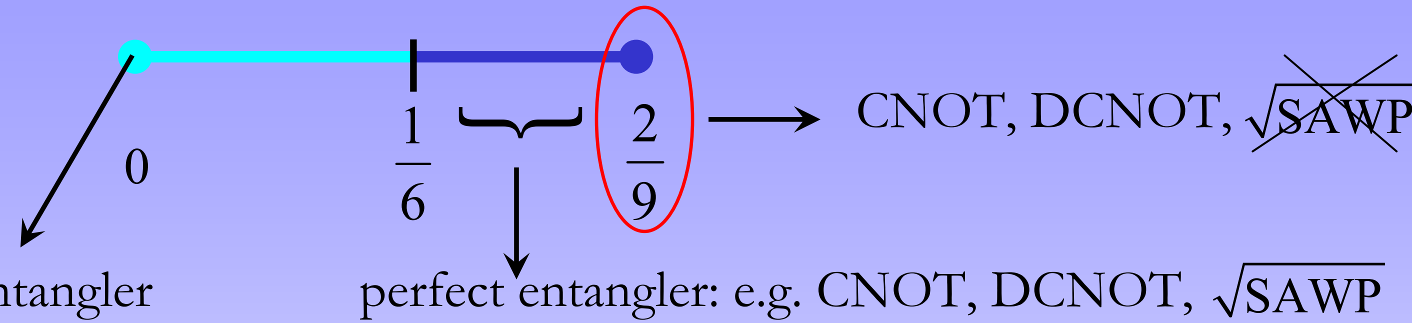
- Special perfect entangler: maximally entangling some full separable basis ?

  E.g.: CNOT, DCNOT, ~~$\sqrt{\text{SAWP}}$~~



the only braiding operators that are perfect entanglers:
CNOT, DCNOT

$$e_p(U) = \frac{1}{18}[3 - (\cos 4c_1 \cos 4c_2 + \cos 4c_2 \cos 4c_3 + \cos 4c_3 \cos 4c_1)]$$



Question: Is there any very strong entangler other than CNOT & DCNOT?

If NOT, all strongest possible entanglers are braid entanglers. ☺

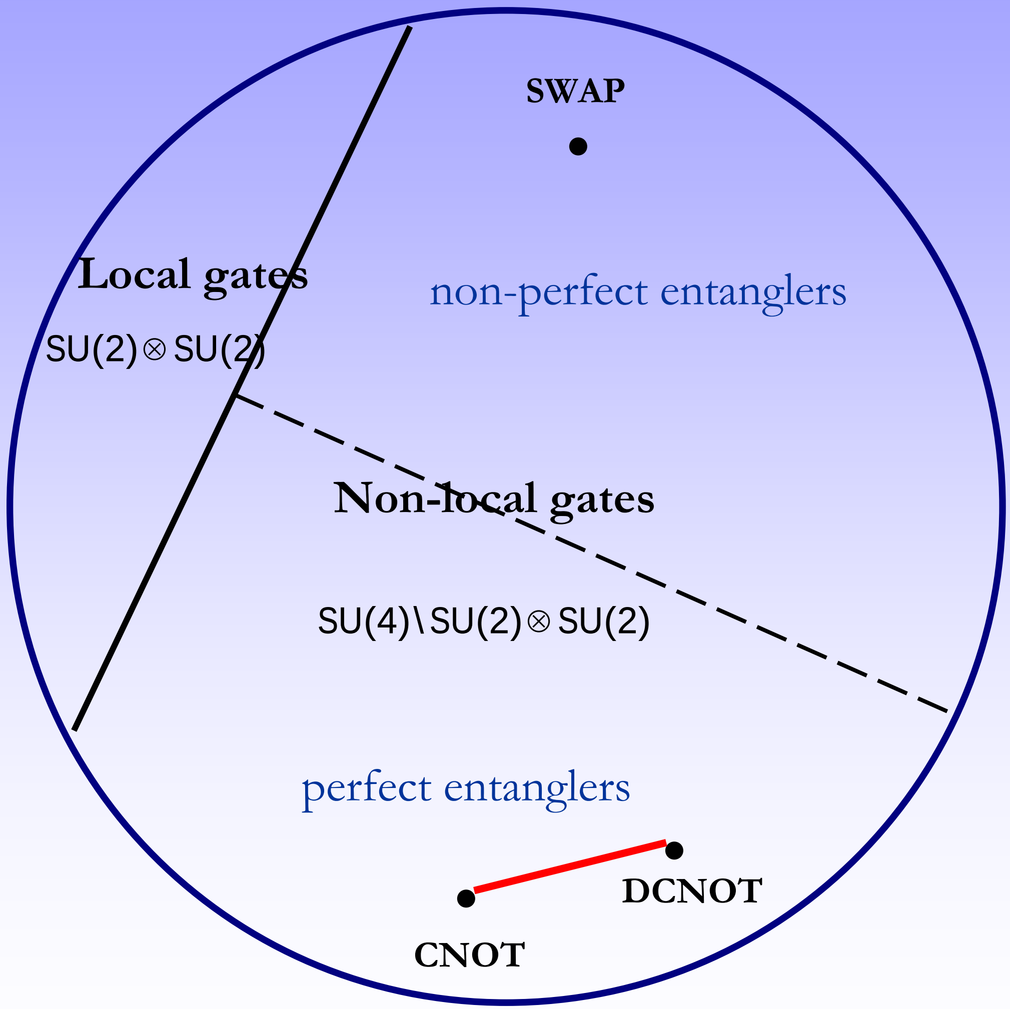
Answer: **Theorem:**

$$e_p(U) = e_{p_{\max}} = \frac{2}{9} \Longleftrightarrow U \equiv [\frac{\pi}{4}, \varphi, 0] \Longleftrightarrow U : \text{special perfect entangler}$$

CNOT $\quad 0 \le \varphi \le \frac{\pi}{4} \quad$ DCNOT

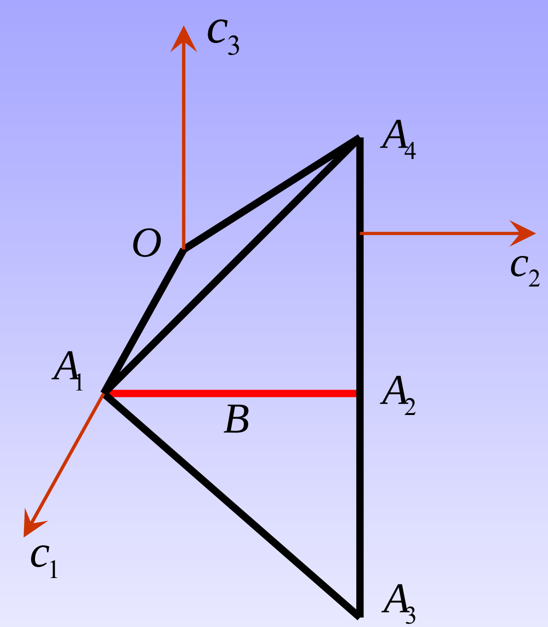
$$C[\varphi] := e^{-i(\frac{\pi}{4}\sigma_x \otimes \sigma_x + \varphi \sigma_y \otimes \sigma_y)}$$

Geometric picture:



$$\frac{\pi}{4} \geq c_1 \geq c_2 \geq |c_3|$$

the Weyl chamber



$$A_1([\frac{\pi}{4},0,0]) \equiv \text{CNOT}$$

$$A_2([\frac{\pi}{4},\frac{\pi}{4},0]) \equiv \text{DCNOT}$$

$$B([\frac{\pi}{4},\frac{\pi}{8},0])$$

Then, braiding entanglers are not the only possible strongest entanglers!

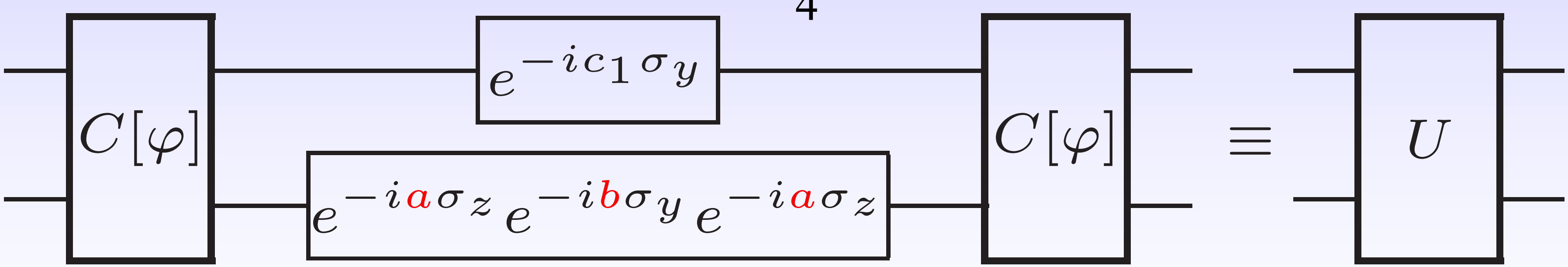
further investigation: importance of special perfect entanglers ?

their special role in universal gate construction

By using at least **3** CNOTs or DCNOTs any unitary gate can be constructed or simulated,

while: only **2** applications of other special perfect entanglers is sufficient!

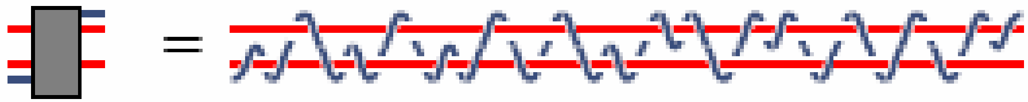
$$0 < \varphi < \frac{\pi}{4}$$



→ minimal universal gate simulation → *B*-gate

Some related subjects :

- an improvement: Universal TQC with only one mobile quasiparticle (Simon et al.)



- Fault-tolerant one-way QC with cluster states (Raussendorf et al.)
- a quantum algorithm calculation of the **Jones polynomials**: a BQP-complete problem (Freedman, Kitaev, Aharonov, et al.)
- Quantum/Topological order: root of order in FQH states (Wen)

  topological degeneracy of the ground state of Kitaev's model

  relation of topological order and entanglement of ground state (Kitaev & Preskill)

  braid group analysis of topological order (Sato et al.)

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